Irreducible representation of GL(2,K) where K is a finite field

Utsav Choudhury

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Contents

Chapter 1

Introduction

In this report I have tried to briefly describe the irreducible representations of Gl(2,K) where K is a finite field .I have not given proofs of those results which uses regular tricks and calculations, for some results I have given the main idea of the proof . Mainly I have followed Ilya Piatetski-Shapiros notes on Complex Representations of GL(2,K) for finite fields K .

1.1 Representations

Let V be a finite dimensional vector space over the field $\mathbb{C}$ of the comlex numbers.

Definition 1. *A linear representation of G in V is a homomorphism ρ of G into Aut(V).*

V is said to be the representation space of ρ and also denoted by V_ρ . The dimension of of ρ is defined to be the dimension of V_ρ .

A homomorphism θ between two representation V_ρ and $V\rho'$ is a linear transformation such that $\theta \circ \rho(g)v = \rho'(g) \circ \theta v$ for all $g \in G$ and $v \in V$.

Sometimes one dimensional representation of G will be called character in this report.In particular,the unit character is the trivial homomorphism.

If ρ fixes a subspace V of V_ρ , i.e V is invariant under each $\rho(g)$ then ρ gives a representation of G say ρ' in this vector space which is called the subrepresentation of ρ .

A representation ρ of G is said to be irreducible if it does not have any subrepresentation. Schur's Lemma:

Lemma 1. *Let ρ_1 and ρ_2 are two irreducible representations of G,and f be a homomorphism between V_ρ and V'_ρ,Then*

(1) If ρ_1 and ρ_2 are not isomorphic then f=0.

(2) If $V_{\rho_1} = V_{\rho_2}$ and $\rho_1 = \rho_2$, f is a scalar multiple of the identity .

Theorem 1. *Every representation ρ of G can be written as direct sum $\rho = \oplus_{i=1}^{k} n_i\rho_i$ where ρ_i are distinct irreducible representations of G.This decomposition is unique,upto the order of the summands.*

Theorem 2. *The number of irreducible representaions of G is equal to the number of conjugacy classes of G.*

The dimensions of all the irreducible representations of G satisfy the formula $\sum_{i=1}^{n}(dim\rho_i)^2 = |G|$. This implies that the irreducible representations of G are dimension 1 and that their number is equal to $|G|$.The set of charecters of G forms a multiplicative group $\hat{G}$ which is isomorphic to G.

Artins lemma:

Lemma 2. *If a_χ are complex numbers such that $\sum_{\chi\in\hat{G}}\chi(g) = 0$ for every $g \in G$, then $a_\chi = 0$ for all $\chi \in \hat{G}$.*

Now G is cannonically isomorphic to the dual $\hat{\hat{G}}$ of $\hat{G}$. Hnece dual to this lemma is also true: If b_g are complex numbers such that $\sum_{g\in G} b_g\chi(g) = 0$ for every $\chi \in \hat{G}$, then $b_g = 0$ for all $g \in G$.

If g is a any finite group then it has $(G : G')$ characters , where G' is the commutator subgroup of G.

If ρ and ρ' are two representation of G then we write $(\rho, \rho') = dimHom_{C[G]}(V_\rho, V'_\rho)$. An irreducible representation ρ appears in a representation ρ' iff $(\rho, \rho') \neq 0$. (ρ, ρ') is then the multiplicity of ρ in ρ'.

Let $End_{C[G]}V_\rho = Hom_{C[G]}(V_\rho, V_\rho)$. Then $End_{C[G]}V = \oplus M_{n_i}(C)$. Hence $(\rho, \rho) = \sum n_i^2$. It follows that ρ has no multiple components iff $End_{C[G]}V_\rho$ is commutative.

1.2 Induced representation

Let G be a finite group and let H be a subgroup aperating on a finite dimensional C-vector space W through a representation $\tau : H \rightarrow AutW$.Define a vector space V to be the set of all functions $f : G \rightarrow W$ that satisfy

$f(hg) = \tau(h)f(g)$ for all $h \in H$ and $g \in G$. Define an operation of G on V by (sf)(g) = f(gs) for $s, g \in G$ and $f \in V$. The $C[G]$-module V thus obtained is called the induced module of W from H to G and is denoted by $Ind_H^G\tau$.

Let E be a $C[G]$ module and denote by $Res_H^G E$ the $C[H]$ module obtained from E by considering only the action of H.

Frobenius Reciprocity Theorem:

Theorem 3. *$Hom_{C[G]}(Ind_H^G W, E) \cong Hom_{C[H]}(W, Res_H^G E)$.*

1.3 Mackeys theorem

Theorem 4. *Let H and J be subgroups of a finite group G . Let ρ and σ be representation of H and J respectively . Then $Hom_{C[G]}(Ind_H^G V_\rho, Ind_J^G V_\sigma)$*

is isomorphic to t he vector space of all functions $F : G \to \ Hom_C(V_\rho, V_\sigma)$ *satisfying*

$F(jgh) = \sigma(j) \circ F(g) \circ \rho(h)$. *for all* $j \in J$,$g \in G$ *and* $h \in H$

when H=J and $\rho = \sigma$ then $Hom_C[G](Ind_H^G V_\rho, Ind_J^G V_\sigma) = End_C[G](V_{r\hat{h}o})$.

If X be the space of functions satisfying 1 then F gets structure of an algebra . If $F_1, F_2 \in X$ then the porduct is given by $(F_1 \star F_2)(g) = \frac{1}{(G:H)} \sum_{s \in G} F_1(gs^{-1})F_2(s)$.

1.4 The group GL(2,K)

Let B = $(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in K^\times \ ; \beta \in K)$.

and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|B| = (q-1)^2 q$ and (G:B) = q+1. Let U = $(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in K)$.

D = $(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in K^\times)$.

U is isomorphic to additive group K and D is isomorphic to $K^\times \times K^\times$. U is the commutator subgroup of B and B is the semi-direct porduct of U by D.In particular $(B : U) = (q-1)^2$ so B has exactly $(q-1)^2$ characters. Another important subgroup of B of order (q-1)q and of index (q-1) in B is

P = $(\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha \in K^\times, \beta \in K)$.

The center Z = $(\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in K^\times)$.

B is the semi direct porduct of Z and P. U is contained in P and it is the commuatator subgroup of P. Let A = $(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in K^\times)$.

A is isomorphic to $K^\times$ and P is the semidirect porduct of U by A.

We will use bruhat dicomposition of G, namely $G = B \uplus BwU$. If $\gamma \neq 0$, then

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta - \alpha\gamma^{-1}\delta & \alpha \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix}$.

Let w' = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, z = $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = w'z$ Then we have the following relations between w' and the elements of B: $w' \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} w'^{-1} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$,(1)

$w'^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, (2)

$s^3 = 1$. (3)

Proposition 1. *GL(2,K) is the free group generated by B and w' with 1, 2, 3 as the definig relations.*

1.5 Conjugacy classes of Gl(2,K)

Let g is any element of Gl(2,K).Then it's eigenvalues are the solution of the equation det(g - XI) = 0.

(a) The eigen values of G belong to K :

If they are equal g is conjugate over K to a unique matrix of the form $c_1(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, or $c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ depending on whether the minimal polynomial of g is different from the characteristic polynomial or equal to it.If the eigenvalues are unequal and it is α, β then the jordan form is $c_3(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

There are (q-1) matrices of the form $c_1(\alpha)$, (q-1) matrices of the form $c_2(\alpha)$ and $\frac{1}{2}(q-1)(q-2)$ of the form $c_3(\alpha, \beta)$.

B) The eigenvalues of g do not belong to K.

Then g is conjugate to $c_4(\alpha) = \begin{pmatrix} 0 & -N\alpha \\ 1 & Tr\alpha \end{pmatrix}$ There are $\frac{1}{2}(q^2 - q)$ matrices of the form $c_4(\alpha)$.

Chapter 2

Inducing Representation from B

In this chapter the irreducible representation of G that are induced from the representation of B (specially the one dimensional representation) are discussed.

2.1 Representation of P

Let ψ be a non-unit character of K^+ . As K^+ is isomorphic to U we can consider ψ as a non-unit charecter of U.

Now for every a $\in$ A we can define a charecter ψ_a of U by

$$\psi_a(u) = \psi(aua^{-1}), for\ u \in U \tag{2.1}$$

If $a \neq a'$,then $\psi_a \neq \psi_{a'}$. This will give distinct q-1 representation of U,and together with the unit representation these are all the representation of U.(as U is abelian and $|U| = q$)

Every character χ of A will give a charecter $\tilde{\chi}$ of P defined by $\tilde{\chi}(ua) = \chi(a)$.This are the all charecters of P as the number of this distinct charecters is q-1 and $(P : U) = q - 1$ and U is the commutator subgroup of P.

To get the higher dimensional representation of P we will induce ψ from U to P .

$$Res^P_U Ind^P_U \psi = \bigoplus_{a \in A} \psi_a. \tag{2.2}$$

Theorem 5. *The group P has q irreducible representations :*

(a) (q-1) of them are one dimensional ; they are of the form $\tilde{\chi}$;

(b) one (q-1)-dimensional representation which is $\pi = Ind^P_U \psi$.

Proof. part (a) is proved earlier . To prove part (b) first observe that $Ind^P_U = (P : U) = q - 1$.By the Frobenius reciprocity theorem and by (2.2)

$$(Ind^P_U \psi, Ind^P_U \psi)_P = (\psi, \bigoplus_{a \in A} \psi_a)_U = 1 \tag{2.3}$$

So Ind_U^P is an irreducible representation of P. Now

$$\sum_{a \in A} (dim\psi_a)^2 + (dimInd_U^P\psi)^2 = (q-1) + (q-1)^2 = |P|; \tag{2.4}$$

Hence there are no more irreducible representation.

Proving (b) without using finiteness of P:

$$Res_U^P Ind_U^P \psi_{a'} = \bigoplus_{a \in A} \psi_a. \tag{2.5}$$

By the earlier technique we can prove thet $Ind_U^P\psi_{a'}$ is irreducible moreover using frobenius reciporcity and by (2.5) $Ind_U^P\psi = Ind_U^P\psi{-}a'$. Now let σ is an arbitrary representation of P. For every $a' \in A$ consider

$$(\sigma, Ind_U^P\psi) = (Res_U^P\sigma, \psi_{a'}) = K; \tag{2.6}$$

Either $K > 0$ then $\sigma = Ind_U^P\psi$ (since both are irreducible).Else $Res_U^P\sigma$ doesnot contain any one dimensional representation of the form ψ_a hence it is the multiple of the unit charecter of U.As A is abelian $Res_A^P V_\sigma$ decomposes into linear A subspaces so there exist a nonzero vector $v \in V_\sigma$ and a charecter χ of A such that $\sigma(a)v = \chi(a)v$ for every $a \in A$. Now $\sigma(u)v = v$ hence $\sigma(ua)v = \chi(a)v$. Hence $\sigma = \tilde{\chi}$.

□

2.2 Representations of B

In this section we will use the properties of group B described in chapter 1 to determine the irreducible representation of B.In the case of B the irreducible representation somehow directly comes out from the representations of its good subgroups such as D,U,Z and P.First of all as D is isomorphic to $K^\times \times K^\times$ every pair (μ_1, μ_2) of charectesr of $K^\times$ will give a unique charecter μ of D given by $\mu \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \mu_1(\alpha)\mu_2(\beta)$.Now $B|U$ is isomorphic to D hence we can extend μ to get a unique charecter of B which is given by $\mu \begin{pmatrix} \alpha & \star \\ 0 & \beta \end{pmatrix} = \mu_1(\alpha)\mu_2(\beta)$ (1). Similarly given any charecter of B as U is the commutator subgroup of B we can get a charecter of $B|U$ (which is isomorphic to D) and hence a pair of unique charecters (μ_1, μ_2) of $K^\times$. This way we get $(q-1)^2$ distinct charecters of B which are all the charecters.Now $w \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} w^{-1} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$. Now let for every charecter μ of B given by (1) we can get a charecter μ_w of B by $\mu_w \begin{pmatrix} \alpha & \star \\ 0 & \beta \end{pmatrix} = \mu_2(\alpha)\mu_1(\beta)$. Clearly $\mu_w(d) = \mu(wdw^{-1})$ for $d \in D$ and $\mu_w = \mu$ iff $\mu_1 = \mu_2$.

B is also direct product of Z by P . Hence the q - 1 charecters of Z can be lifted to distinct charecters of B by

$\tilde{\chi}(zp) = \chi(z)\ z \in Z, p \in P$ where χ is a charecter of Z .

Also from the irreducible representation π of P we can get a irreducible representation $\tilde{\pi}$ fo dimension q-1 : $\tilde{\pi}(zp) = \pi(p)\ z \in Z,\ p \in P$. (2)

Tensoring $\tilde{\chi}$ with $\tilde{pi}$ we get q -1 dimensional representation of B whose restriction to Z is χ. $(\tilde{\chi} \otimes \tilde{\pi})(zp) = \chi(z)\pi(p)\ z \in Z,\ p \in P$. We get q-1 distinct (q-1)-dimensional irreducible representation of B.

Theorem 6. *The group has*

(a)$(q-1)^2$ charecters given by (1)

(b)q-1 (q-1)-diemnsional irreducible representation given by (2)

Proof. By counting arguement one can prove that this are all the representations.

(proof without using finiteness of B) Suppose ρ is a irreducible representation of B.

Case 1:

ρ is a charecter : $\rho(zp) = \rho(z)\rho(p)\ z \in Z\ p \in P$ now P is semidirect product of U by A and U is the commutator subgroup of B hence $\rho(u) = 1\ u \in U$.

$$\rho(zp) = \rho(z)\rho(p) = \rho(z)\rho(u)\rho(a) = \rho(z)\rho(a). \tag{2.7}$$

where $p = ua$ for some $u \in u\ a \in A$. Now $\begin{pmatrix} \alpha & \star \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^{-1}\alpha & 0 \\ 0 & 1 \end{pmatrix}$ now $\rho(u) = 1$ so $\rho \begin{pmatrix} \alpha & \star \\ 0 & \beta \end{pmatrix} = \rho \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

hence ρ is of the form (1).

Case 2 : ρ is higher dimensional. Now Z is the center of B hence $z \in Z$ commutes with ever element of B .So by schurs lemma the linear transformation $\rho(z)$ is multiplication by constant .Now restriction of ρ to P is a irreducible representation of P.Otherwise there will be a P invariant subspace $V \subset V_\rho$ and if $v \in V$ then $\rho(zp)(v) = \rho(z)\rho(p)(v) \in V$ .Hence V is a B invariant subspace,but ρ is irreducible . So ρ is of the form (2).

□

2.3 Inducing charecters from B to G

Let $\hat{\mu} = Ind_B^G \mu$ where μ is a charecter of B.

Definition 2. *Given a representation ρ of G the Jacquest Module $J(V_\rho)$ is a subspace of V_ρ defined as $J(V_\rho) = (v \in V_\rho | \rho(u)v = v\ for\ every\ u \in U)$.*

U is normal in B hence B acts on $J(V_\rho)$. If ρ_1 and ρ_2 are two representations of G then $J(V_{\rho_1} \oplus V_{\rho_2}) = J(V_{\rho_1}) \oplus J(V_{\rho_2})$.

Lemma 3. *If μ is a charecter of B , then dim $J(V_{\hat{\mu}}) = 2$.*

Proof. $J(V_{\hat{\mu}})$ consistes of all functions $f : G \to C$ that satisfies $f(bg) = \mu(b)f(g)$ and f(bu) = f(b). Now using Bruhats decomposition we can porve that f is determined by its values in 1 and w.Indeed we can give a basis f_1, f_2 satisfying $f_1(1) = 1$ and $f_1(w) = 0$

$f_2(1) = 0$ and $f_2(w) = 1$.

□

Lemma 4. *If μ is a charecter of B, then B operating on $J(V_{\hat{\mu}})$ has two eigen vectors $f_1,\ f_2$ defined by (1) that correspond to the eigenvalue μ and μ_w respectively. $\hat{\mu}(b)f_1 = \mu(b)f_1$ and $\hat{\mu}(b)f_2 = \mu_w(b)f_2$ for every $b \in B$.*

Proof. Here we have to prove that both sides of the equalities coincide in 1 and w.Though all the other cases are easy to check the only trick here is to compute $(\hat{\mu}(b)f_2)(w)$.(details are given in Shapiro). □

Lemma 5. *Let ρ be a representation of G.Then $J(V_\rho) \neq\ 0$ iff there exists a charecter μ of B such that $(\rho, \hat{\mu}) \neq\ 0$.*

Corollary 1. *Let ρ be an irreducibel representation of G.Then $J(V_\rho) \neq\ 0$ iff there exists a charecter μ of B such that $\rho \leq\ \hat{\mu}$.*

Corollary 2. *If μ is a charecter of B then $\hat{\mu}$ has at most two ireducible components.*

Lemma 6. *If μ is a charecter of B and $\mu = \mu_w$,then $\hat{mu}$ has a one dimensional component.*

Proof. first of all $\mu(b) = \mu_1(detb)$.Now define a function $f : G \to C$ by $f(g) = \mu_1(detg)$. then $f \in V_{\hat{\mu}}$.And f is an eigenvector of G that belongs to the eigenvalue $\mu_1 \circ det$. □

Lemma 7. *If μ is a charecter of B, then $\hat{\mu}$ has at most one one dimensional component.*

Lemma 8. *If μ is a charecter of B,then $Res_P V_{\hat{mu}}\ \equiv\ Res_P J(V_{\hat{mu}}) \oplus\ V_\pi$*

Proof. Let V be the P-complement to $J(V_{\hat{\mu}})$ in $V_{\hat{\mu}}$.Then dim V = q -1, and it doesnot contain any one dimensional subspace otherwise U being the commutator subgroup of P will act as identity in that one dimensional subspace which is a contradiction.Hence V does not contain any representation of dimension $< q-1$ .So V is irreducible and V is isomorphic to V_π.

□

Lemma 9. *If μ is a charecter of B and $\hat{\mu}$ is reducible then: (a) $\hat{\mu}$ has one dimensional component. (b) $\mu\ = \mu_w$*

Lemma 10. *If μ and μ' are two distinct charecters of B.Then $(\hat{\mu}, \hat{\mu}') \neq 0$ iff $\mu' = \mu_w$.*

Corollary 3. *Let μ and μ' be two distinct charecters of B then $\hat{\mu} = \hat{\mu}'$ iff $\mu' = \mu_w$.*

Theorem 7. *The irreducible representation of G,which are components of induced representations of the form $\hat{mu}$ where μ is a charecter of B,split up into the following classes:*
a) q-1 one-dimensional representations
b) q-1 q- dimensional representations,
c)$\frac{1}{2}(q-1)(q-2)$ (q+1)-dimensional representations.

Proof. (1) $\mu = \mu_w$,i.e,$\mu_1 = \mu_2$.In this case $\hat{\mu}$ is the direct sum of a onedimensional representation and an irreducible q- dimensional representation.There are (q-1) characters of $K^{\times}$, hence in this way we obtain q-1 charecters of G and q-1 q-dimensional representation. (2)$\mu \neq \mu_w$,,i.e, $\mu_1 \neq \mu_2$¿ In this case $\hat{\mu}$ is an irreducible representation of dimension q+1.The number of such μ is $(q-1)^2-(q-1)$ and further μ and μ_w induce the same representation.Hence we obtain in this way $\frac{1}{2}(q-1)(q-2)$ irreducibel representations of G of dimension q+1.

□

2.4 The dimension of cuspidal representation

Definition 3. *Irreducible representation of G that are not components of $\hat{\mu}$,with μ a character of B,are said to be cuspidial.*

Lemma 11. *a)Let ρ be a cuspidial representation of G then $Res_P\rho = \pi$. b) Conversely,if ρ is a representation of G such that $Res_P\rho = \pi$ then ρ is cuspidial.*

Proof. b)Representation π is irreducible hence ρ is irreducible . $dim\rho = q-1$ .So it cannot be a component of $\hat{\mu}$ as the irreducible component of $\hat{\mu}$ has dimension 1,q or (q-1).Hence ρ is cuspidial.

a) Firstly $Res_P V_\rho$ can not have one dimensional component otherwise there will be a nonzero vector $v \in V_\rho$ and a charecter χ of P such that $\rho(p)v = \chi(p)v$ for evey $p \in P$.In particular $\rho(u)v = v$ for every $u \in U$.Hence $J(V_\rho) \neq 0$,but ρ is cuspidal .So $Res_P\rho$ is a multiple of π.

Let $\hat{\pi} = Ind_B^G\pi = Ind_U^G\psi$ hence by Mackey's theorem $End_{C[G]}V_{\hat{pi}}$ is isomorphic to the algebra A of all functions F: G $\rightarrow$ C satisfying $F(u_1gu_2) = \psi(u_1u_2)F(g)$ for $u_1, u_2 \in U$ *and* $g \in G$. The multiplication between two functions $F_1/, F_2$ is given by $(F_1 * F_2)(g) = \frac{1}{G:U}\sum_{s\in G} F_1(gs^{-1})F_2(s)$.We will show that A is abelian hence $\hat{\pi}$ does not have any multiple component. Now $(Res_P\rho, \pi) = r$ hence by the forbenius reciprocity $r = (\rho, \hat{\pi}) = 1$. Let us define an involution in A .For that we will define a involution on G: If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then $g' = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}$. Now $(g_1g_2)' = {g_1}'{g_2}'$, $g'' = g$ *and* $g = g'$ if g is symmetric with respect to the second diagonal. For $F \in A$ let us define F': G $\rightarrow$ C by $F'(g) = F(g')$.Clearly $F' \in A$.If we can show that F and F' coincide on representative of the double classes of $U\backslash G/U$.By bruhat's decomposition $G = B \biguplus BwU$ and using $B = UD$ we can verify that the representatives are of the form (a)$\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ or (b)$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$. As F and F' conincide on the matrices of type (a) and on the matrices of type (b) in case $\alpha = \beta$ the work remaining is to prove when $\alpha \neq \beta$. First of all $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\delta\beta \\ 0 & 1 \end{pmatrix}$, If we apply F(or F') on both sides of the above identity we get either $F\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = 0$ or $\psi(\beta) = \psi(\beta\delta\alpha^{-1})$.But the later one will imply $\psi(\beta(1-\alpha^{-1}delta)) = 1\ for\ every\ \beta \in K$ or more precisely ψ is a unit charecter of K^+,which is a contradiction.

□

2.5 Assigning cuspidal representation to non-decomposable charecters of $L^\times$

Let L is the unique quadratic extension of K.If $\alpha \in L$,then $\bar{\alpha}$ denotes its unique conjugate over K.Norm map N : L → K is defined by $N(\alpha) = \alpha\bar{\alpha}$.It is multiplicative and is a homomorphism between $L^\times$ and $K^\times$.By counting the number of elements of ker N (N viewed as homomorphism between $L^\times$ and $K^\times$) it is clear that N is surjective and $N(\alpha) = \alpha\alpha^{q+1}$ (this comes from the fact that the galois group of L over K is generated by the Frobenius automorphism $\alpha \to \alpha^q$).

Corollary 4. *If β is an element of $L^\times$ such that $N(\beta) = 1$, then there exists an $\alpha \in L$ such that $\alpha\bar{\alpha}^{-1} = \beta$*

Definition 4. *Let χ be a charecter of $K^\times$.Composing χ with the norm map $L^\times$ to $K^\times$, we obtain a charecter $\tilde{\chi}$ of $L^\times$.*
$\tilde{\chi}(\alpha) = \chi(N\alpha)\ \alpha \in\ L^\times$ (1).
$\tilde{\chi}$ is said to be decomposible.

If ν is an arbitrary charecter of $L^\times$, then $\bar{\nu}$ its conjugate over K , $m\bar{u}(\alpha) = \mu(\bar{\alpha})$.

Lemma 12. *A charecter ν of $L^\times$ is decomposible iff $\nu = \bar{\nu}$.*

Lemma 13. *If μ is a nondecomposable charecter of $L^\times$, then $\sum_{Nx=\alpha} \nu(x) = 0$ for every $\alpha \in K^\times$.*

Let Tr : $L^+ \to K^+$ be the map defined by $Tr(x) = x + \bar{x}$. Clearly trace is linear map and moreover trace is surjective .

Corollary 5. *If $\alpha \in L^\times$, then for every $\beta \in K$ there exists an $x \in L$ such that $\alpha x + \bar{\alpha}\bar{x}\ = \beta$.*

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Let ν be a nondecomposable charecter .To define a cuspidal representation of ρ of G we first take the (q-1) dimensional vector space of all functions f: $K^{\times} \to$ C.After that we will define ρ on P such that it equals to π on P. $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$.

Identifying subgroup A of P with $K^{\times}$ we are led tro the following defenition: $(\rho \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} f)(x) = \psi(\beta x) f(\alpha x)$.

Also to make ρ coincide with ν on Z: $(\rho \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} f)(x) = \nu(\delta) f(x)$.

As B is semi direct product of Z and P we have to define ρ on B by $(\rho \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} f)(x) = \nu(\delta)\psi(\beta\delta^{-1}x) f(\alpha\delta^{-1}x)$. (*)

As defined in 1.4 Gl(2,K) is the free group generated by B and w' with some relation. So we have to now define ρ on w' adn check that those relations are satisfied. To do so we define a function j : $K^{\times} \to$ C by $j(u) = \frac{1}{q} \sum_{\substack{Nt=u \\ t\in L^{\times}}} \psi(t + \bar{t})\nu(t)$

j satisfies the following two identies whose proofs are given in Ilya Piatetski-Shapiro's notes.

1)

$$\sum_{v\in K^{\times}} j(uv)j(v)\nu(v^{-1}) = \begin{cases} \nu(-1) & u = 1 \\ 0 & u \neq 1 \end{cases}$$

2) $\sum_{v\in K^{\times}} j(xv)j(yv)\nu(v^{-1})\psi(v) = \nu(-1)\psi(-x-y)j(xy)$.

Assuming above identities we define ρ on w' by: for any f : $K^{\times} \to$ C $(\rho(w')f)(y) = \sum_{x\in K^{\times}} \nu(x^{-1})j(yx)f(x)$.

Using 1) and 2) it can be shown that $\rho(w')$ together with (*) is compatible with the identitites decribed in 1.4.(for details see Shapiro's notes).

We can therefore calculate $(\rho(g)f)(y)$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ by writing $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta - \alpha\gamma^{-1}\delta & -\alpha \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix}$.

$(\rho(g)f)(y) = \sum_{x\in K^{\times}} k(y,x;g)f(x)$.*

where
$k(y,x;g) = \frac{1}{q}\psi(\frac{\alpha y+\delta x}{\gamma} \sum_{\substack{u\bar{u}=yx^{-1}det(g) \\ u\in L^{\times}}} \psi(-\frac{x}{y}(u+\bar{u}))\nu(u)$. **

we will denote this ρ by ρ_ν

Proposition 2. *(a)If ν is a non-decomposable charecter of $L^{\times}$, then the representation ρ_ν is cuspidal.*

(b)If ν and ν' are non-decomposable charecter of $L^\times$,then ρ_ν is isomorphic to ρ'_ν iff ν is a conjugate to ν' over K.

Proof. (a) As ρ_ν equals to π on P ρ_ν is cuspidal.

(b) Let $\rho_\nu = \rho$, $\rho'_\nu = \rho'$, $j_\nu = j$ and $j'_\nu = j'$. If ν' is conjugate to ν then j = j'.Hence $\rho = \rho'$.

Conversely suppose ρ' is isomorphic to ρ then there exist an nonsingular linear transfromation θ of the space of all comlex valued function of $K^\times$ say V such that it commutes with $\rho(g)$ for $g \in G$ in particular it commutes with $\rho(p)$ for $p \in P$ but $Res_P\rho$ and $Res_P\rho'$ are both equlats to π hence by schurs lemma θ is multplication by a scalar and as $\rho'(g) = \theta\rho(g)\theta^{-1}$ implies $\rho'(g) = \rho(g)$ for $g \in G$.

Hence

$$\nu'(\delta)f(\delta{-}1y) = (\rho' \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} f)(y) = (\rho \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} f)(y) = \nu(\delta)f(\delta^{-1}y). \tag{2.8}$$

Hence $\nu'(\delta) = \nu(\delta)$ $\delta \in K^\times$ (1).

Again $\rho'(w') = \rho(w')$ gives

$$\sum_{x \in K^\times} \nu'(x^{-1})j'(yx)f(x) = \sum_{x \in K^\times} \nu(x^{-1})j(yx)f(x) \tag{2.9}$$

for every $y \in K^\times$ and every $f \in V$.2.9 and 1 implies that j'(u) = j(u) for every $u \in K^\times$.This implies

$$\sum_{\substack{Nt=u \\ t \in L^\times}} \psi(t + \bar{t})\nu'(t) = \sum_{\substack{Nt=u \\ t \in L^\times}} \psi(t + \bar{t})\nu(t). \tag{2.10}$$

Let $\delta \in K^\times$ .Replacing the variable t in 2.10 by δt and using one to cancel $\nu(\delta)$ on both sides and arranging the terms we can rewrite 2.10 as

$$\sum_{t\bar{t}=v}{}' (\nu'(t) + \nu'(\bar{t}) - \nu(t) - \nu(\bar{t}))\psi(\delta(t + \bar{t})) = 0. \tag{2.11}$$

Where the prime over the summation symbol indicates that for every $t \in L^\times$ such that $t\bar{t} = v$, only one pair out of the two $(t, \bar{t}$, $(\bar{t}, t)$ contributes to the summation. Now given any x in K.Cinsider the quadratic equation $X^2 - xX + v$ .The solution t of this equation belongs to L and $t + \bar{t} = x$ and $t\bar{t} = v$. Hence we can write $a_x = \nu'(t) + \nu'(\bar{t}) - \nu(t) - \nu(\bar{t})$.

and 2.11 can be written as $\sum_{t\bar{t}=v} a_x\psi(\delta x) = 0$.

If $x_1 \neq x_2$ the charecters $\delta \to \psi(\delta x_i)$ of K^+ are distinct hence by artins lemma $a_x = 0$ for all x in K. this implies $\nu'(t) + \nu'(\bar{t}) = \nu(t) + \nu(\bar{t})$ $t \in L^\times$ and by 1 $\nu'(t)\nu'(\bar{t}) = \nu(t)\nu(\bar{t})$. Hence the pairs $(\nu'(t), \nu'(\bar{t}))$, $(\nu(t), \nu(\bar{t}))$ are the solutions of same quadratic equations over complex numbers. It is true for a generator t of the cyclic group $L^\times$ .So either $\nu(t) = \nu'(t)$ which implies $\nu(t') = \nu'(t')$ for every t in $L^\times$ or $\nu(\bar{t}) = \nu'(\bar{t})$ hence ν' is conjugate to ν over K.

$\square$

Chapter 3

Bessel functions and Γ functions

In this chapter we will briefly discuss Whittaker model and uniqueness of it , Bessel functions , Γ-functions.

3.1 Whittakers models

If χ is a charecter of G, then by frobenius reciprocity theorem $(\chi, Ind_U^G \psi)_G = (\chi, Ind_B^G \pi) = (Res_B \chi, \pi)_B = 0$.

Hence all irreducible components of $Ind_U^G \psi$ are of dimension ¿ 1.

Lemma 14. *Let ρ be a representation of G of dimensionr ¿ 1. Then:*

(a) $Res_P V_\rho = Res_P J(V_\rho \oplus V_\pi$.

(b) $dim J(V_{rho}) = dim \rho - (q-1)$.

(c) The multiplicity of ρ in $Ind_U^G \psi$ is equal to 1.

Proof. If $dim\rho = q-1$, then ρ is cuspidal, then $J(V_\rho) = 0$ and $Res_P V_\rho = V_\pi$. If $dim\rho \geq q$' then ρ is noncuspidal,and $J(V_\rho \neq 0$. and $\hat{\mu} = \rho' \oplus \rho$ and $dim\rho' \leq 1$. Now $Res_P V'_\rho \oplus Res_P V_\rho = Res_P J(V'_\rho) \oplus Res_P J(V_\rho) \oplus V_\pi$. (1)

Now V_π doesnot contain any one dimensional P-moduole .So $V_\pi \subseteq Res_P V_\rho$ and $Res_P J(V_\rho) \subset V_\rho$. Moreover $J(V_\rho)$ decomposes into a direct sum of one dimensional P-subspaces.Hence $Res_P V_\pi \cap Res_P J(V_\rho) = 0$. So the right hand side of (a) is contained inside the left hand side.If $dim\rho = q$, then since the dimension of the right hand side of a $\geq 1 + (q-1)$ so the equality holds. If $dim\rho = q+1$, then $\rho' = 0$ and 1 coincides with (a).We have also proved thet multiplicity of π in $Res_P \rho$is 1. So using frobenius recoprocity theorem the multiplicity of ρ in $Ind_U^G \psi$ is 1. ∎

If ρ is a irreducible higher dimensional representation of G,then V_ρ can be embedded in $Ind_U^G V_\psi$.For every $v \in V_\rho$ there exists a function $W_v : G \to C$ called a Whittaker function of ρ which satisfies the following property:

$W_v = 0 \Leftrightarrow \ v = 0,$

$W_{c_1 v_1 + c_2 v_2} = c_1 W_{v_1} + c_2 W_{v_2}$ for $c_1, c_2 \in C$

$W_v(ug) = \psi(u) W_v(g) \ \ for \ u \in U \ g \in G.$

$W_{\rho(s)v}(g) = W_v(gs) \ \ for \ s, g \in G.$

The set of all functions W_v form a G-subspace $W(\rho)$ of $Ind_U^G V_\psi$ called the Whittaker model of ρ.By previous lemma this subspce is uniquely determined within $Ind_U^G V_\psi$.

3.2 The Γ-function of a representation

Let ρ be a higher dimensional representation of G.We have seen that the subgroup A of P is isomorphic to $K^\times$ and the P decomposition $V_\rho = J(V_\rho) \oplus V_\pi$ is also an A decomposition . If $dim\rho \geq q$, then $\rho = \rho_{\mu_1,\mu_2}$, where μ_1 and μ_2 are charecters of $K^\times$.Now by lemma 2 μ and μ_w are the eigenvalues of A on $J(V_\rho$.If $dim\rho = q+1$, then $\mu \neq \mu_w$ and $dimJ(V_\rho) = 2$.If $dim\rho = q$, then $\mu = \mu_w$ and $dimJ(V_\rho) = 1$ .If $dim\rho = q-1$ then $J(V_rho) = 0$.In the first two cases we call μ_1^{-1},μ_2^{-1} and μ_1^{-1}, respectively , the exceptional charecters for ρ.

Lemma 15. *If a charecter ω of $K^\times$ is not a exceptional charecter for ρ, then any two linear functionals l_1, l_2 of V_ρ satisfying*

$l_i(\rho \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} v) = \omega(x)^{-1} l_i(v)$ *for every $x \in K^\times$ and $v \in V_rho$ are linearly independent.*

Theorem 8. *If ρ be a higher dimensional irreducible representation of G and let ω be a charecter of K^{times} which is not exceptional for ρ . Then there exists a complex number $\Gamma_\rho(\omega)$ such that for evrey Whittaker function W_v of ρ we have $\Gamma_\rho(\omega) \sum_{x \in K^\times} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \omega(x) = \sum_{x \in K^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x)$.*

Proof. Define linear functionals l_i,i=1,2 of V_ρ by

$l_1(v) = \sum_{x \in K^\times} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \omega(x)$

$l_2(v) = \sum_{x \in K^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x)$

It is easy to verify that l_i satisfy the porperty described in the previous lemma.Hence l_2 is a multiple of l_1 by a constant . We denote this constant by $\Gamma_\rho(\omega)$.

□

3.3 Determination of ρ by Γ_ρ

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Let ρ be a higher dimensional representation of G. For every $v \in V_\rho$,let W_v be the corresponding Whitaker function of ρ and let r be the homomorphism of of V_ρ into the space $F(K^\times, C)$ of all functions $\varphi : K^\times \to C$, defined by $r(v) = Res_A W_v$. r is an A- homomorphism where the action of $K^\times$ on $F(K^\times, C)$ is given by $(\alpha \cdot \varphi)(x) = \varphi(x\alpha)$.

Lemma 16. *The homomorphism r is surjective and* $Ker\ r = J(V_\rho)$.

The restriction of ρ to Z can be identified with a charecter ω_ρ called the central character of ρ.

Proposition 3. *A cuspidal representaion ρ of G is uniquely determined by its Γ-function and its central character.*

Proof. Let ρ and ρ' be two cuspidal representations of G. Then $Res_P \rho = \pi = Res_P \rho'$ .If ρ and ρ' coincide on Z, then they coincide on B since B=ZP.Suppose $\Gamma_\rho = \Gamma'_\rho$.By bruhat decomposition to show $\rho = \rho'$ we have to actually show $\rho(w) = \rho'(w)$. Without loss of generality we can assume that $Res_A W_v = Res_A {W_v}'$,where $W_v, {W_v}'$ are the whittaker funtion of ρ and ρ'.(using the fact that r is an isomorphism in this case). Therefore by theorem * of the previous section the assumption $\Gamma_\rho = \Gamma'_\rho$ for every charecter ω of $K^\times$,

$\sum_{x \in K^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x) = \sum_{x \in K^\times} {W_v}' \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x)$.

This implies by Artins lemma that $W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = W'_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$.

Hence $W_{\rho(w)v} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = W'_{\rho'(w)v} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ Hence by lemma 16 $W_{\rho(w)v} = W'_{\rho'(w)v}$ and hence $\rho(w) = \rho'(w)$.

□

3.4 The bessel function of a representation

Let ρ be a higher dimensional representation.then $dimV_\rho \;\; > 2 \geq dimJ(V_\rho$ except in the case where q = 3 and $dim\rho = 2$ .In this case ρ is cuspidal and $J(V_\rho = 0$. Therefore $J(V_\rho) \neq V_\rho$ in all cases.

$Res_U\rho$ decomposes into a direct sum of charecters . One of them must be different from the unit chrecter otherwise $\rho \begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix} v = v$ for all $\beta \in K$ and all v in V_ρ. But $W_v \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \psi(\alpha\beta) W_v \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ for every $\alpha \in K^\times$

Hence $Res_A W_v = 0$ and hence $v \in J(V_\rho)$ which contradicts the fact that $J(V_\rho \neq V\rho$. Therefore there exists a nonzero vector $v_1{}'$ and a charecter ψ' of K such that $\rho(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix})v_1{}' = \psi'(\beta)v_1{}'$ for every $\beta \in K$. Now there exists $\alpha \in K^\times$ such that $psi'(\beta) = \psi(\alpha\beta)$. $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ 1)

and replacing $v_1{}'$ by $v_1 = \rho \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} v_1{}'$

we get

$\rho \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} v_1 = \psi(\beta) v_1$ 2) for every $\beta \in K$.

this implies $v_1 \notin \; J(V_\rho)$

moreover if $\alpha \in K^\times$, then $\psi(\beta) W_{v_1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \psi(\alpha\beta) W_{v_1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ for all $\beta \in K$.

if $\alpha \neq 1$, then we may conclude that $W_{v_1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = 0$.

v_1 is called a Bessel vector for ρ. If v_2 is an additional Bessel vector for ρ then , by the last observation there exists a $\zeta \in C$ such that $W_{v_1}(a) = \zeta W_{v_2}(a)$ for every $a \in A$ that implies $(v_1 - \zeta v_2) \in J(V_\rho)$.Hence $\rho \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} (v_1 - \zeta v_2) = v_1 - \zeta v_2 = \psi(\beta)(v_1 - \zeta v_2)$ for all $\beta \in K$.Hence $v_1 = \zeta v_2$.So bessel vector is unique upto scalar multiple.Using this vector we define Bessel function $J_\rho : G \rightarrow \; C$ of ρ by

$J_\rho(g) = [W_{v_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]^{-1} W_{v_1}(g)$.

It does not depend on the choice of the Bessel vector.It is a whittaker function for ρ. Also $J_\rho(1) = 1$ and $J_\rho(a) = 0$ if $a \neq 1$ and $a \in A$. (3)

So if a charecter ω of $K^\times$ is nt exceptional for ρ, then
$\Gamma_\rho(\omega) = \sum_{x\in K^\times} J_\rho \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} \omega(x)$. (4)

3.5 Computation of $\Gamma_\rho(\omega)$ for a non-cuspidal and cuspidal representation

Let ρ be a higher dimensional non cuspidal representation then ρ is a component of $\hat{\mu}$.So every elemnt of V_ρ can be viewed as a functions f : G $\rightarrow$ C such that $f(bg) = \mu(b)f(g)$ whre $b \in B$, $g \in G$. Foe every such $f \in V_\rho$ let $W_f : G \rightarrow C$ be teh function defined by $W_f(g) = \sum_{z\in K} f(w \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} g)\psi(z)^{-1}$. (1)

W_f satisfies the last two equalities of Whittaker's function in section 3.1.

this map is a linear map from V_ρ into $Ind_B^G V_\mu$.To show that its kernel is not the whole V_ρ we define a function f_1 by $f_1(b) = 0$ (2)

$f_1(bwu) = \mu(b)\psi(u)$ for $b \in B$ and $u \in U$. So $f_1 \in Ind_U^G V_\psi$ and it satisfies $f_1(gu) = \psi(u)f_1(g)$ (4)

If $dim\rho = q + 1$, then $V_\rho = Ind_B^G V_\mu$.

If $dim\rho = q$, then $\hat{\mu} = \rho \oplus \rho'$,where ρ' is a one dimensional representation.So we can write $f_1 = f' + f$. Where $f' \in V'_\rho$ and $f \in V_\rho$.Now for a fixed $u \in U$ the functions $g \mapsto f'(gu)$ and $g \mapsto \psi(u)f'(g)$ belong to V'_ρ,similarly $g \mapsto f(gu)$ and $g \mapsto \psi(u)f(g)$ belong to V_ρ.Hence using 4 we get $f'(gu) = \psi(u)f'(g)$ for $g \in G$ (5)

But the function f' satisfies $f'(g) = f'(1)\mu_1(detg)$ So 5 and 6 gives $f'(1) = \psi(u)f'(1)$ for every $u \in U$ hence f'(1) = 0 hence f'= 0.So $f_1 = f \in V_{rho}$.It is easy to verify that $W_{f_1} \neq 0$.

f_1 is Bessel vector too as it satisfies 4. Now we can use this bessel vectors to calculate the value of J_ρ and then we calculate $\Gamma_\rho(\omega)$. If a charecter of $K^\times$ ω is not exceptional for ρ then

$\Gamma_\rho(\omega) = \frac{\omega(-1)}{q} \sum_{r\in K^\times} \mu_1(r)^{-1}\omega(r)^{-1}\psi(r) \sum_{s\in K^\times} \mu_2(s)^{-1}\omega(s)^{-1}\psi(s)$.

Let ν be a non-decomposable chrecter of $L^\times$ and let $\rho = \rho_\nu$ be the corresponding cuspidal representation then

$\Gamma_\rho(\omega) = \frac{q^{-1}}{\nu(-1)} \sum_{t\in L^\times} \nu(t)\omega(t\bar{t})^{-1}\psi(t + \bar{t})$.