# Some Contributions to Motives of Deligne-Mumford Stacks and Motivic Homotopy Theory

**Dissertation**

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr.sc.nat)

vorgelegt der
Mathematisch-naturwissenschaftlichen Fakultät
der
Universität Zürich

von

**Choudhury, Utsav**

aus India

**Promotionskomitee**

Prof. Dr. Joseph Ayoub (Vorsitz und Leiter der Dissertation)
Prof. Dr. Andrew Kresch

Zürich, 2013

# ABSTRACT

Some Contributions to Motives of Deligne-Mumford Stacks and Motivic Homotopy Theory

by
Utsav Choudhury

Vladimir Voevodsky constructed a triangulated category of motives to universally linearize the geometry of algebraic varieties. In this thesis, I show that the geometry of a bigger class of objects, called Deligne-Mumford stacks, can be universally linearized using Voevodsky's triangulated category of motives. Also, I give a partial answer to a conjecture of Fabien Morel related to the connected component sheaf in motivic homotopy theory.

Vladimir Voevodsky hat eine triangulierte Kategorie von Motiven konstruiert, um die Geometrie algebraischer Varietäten "universell zu linearisieren". In dieser Dissertation zeige ich, dass auch die Geometrie einer umfangreicheren Klasse von Objekten, nämlich von Deligne-Mumford stacks, mit Hilfe der triangulierten Kategorie Voevodskys universell linearisiert werden kann. Ausserdem gebe ich eine partielle Antwort auf eine Vermutung von Fabien Morel in Bezug auf die Zusammenhangskomponenten-Garbe in motivischer Homotopie-Theorie.

To mishti and dada

## ACKNOWLEDGEMENTS

First of all, I am sincerely grateful to my supervisor Prof. Joseph Ayoub for his guidance during my PhD. This thesis would not exist without his support. I am also indebted to Prof. Andrew Kresch and Prof. David Rydh for their helpful suggestions on algebraic stacks.

Prof. Oliver Röndings and Prof. Luca Barbieri Viale read a preliminary version of this thesis. I take this opportunity to convey my gratitude to them.

I wish to thank Jonathan Skowera, Martin Gallauer Alves de Souza, Simon Pepin Lehalleur, Alberto Vezzani, Andrea Ferraguti, Thomas Preu, Dörte Kreher, Corina Simian, Asieh Parsania, Prosenjit Roy for all the stimulating conversations on mathematics and beyond.

I will be ever grateful to all my teachers from whom I had the oppurtunity to learn till now. Specially Kusum Choudhury Singh, Digbijay Chaudhuri, Asit Sir, Moloy Ganguly, Swami Suparnananda, Nabarun Bhattacharya, Prof. Clare D'Cruz, Prof. Shiva Shankar, Prof. Madhavan Mukund, Prof. Amritanshu Prasad, Dr. Joël Riou, Shane Kelly, V. Jayant..... thank you for sharing your wisdom. Also, I wish to thank the band members of 1G3B and Pujo Shonkha Private limited. The audacity of our journey for last few years still surprises me.

Lastly, Smruti Sanjita Behera should be held responsible for always keeping the humor and love alive in me.

# PREFACE

A commonly applied technique in algebraic geometry is to attach to a given variety $X$ certain invariants in vector spaces, called cohomology. There are many cohomology theories reflecting different structural properties of varieties. The *theory of motives* is an attempt to find a universal way to study all the cohomology theories. It has a geometric side and a linear algebra side. On the geometric side, the aim is to construct a universal cohomology theory from cycles. On the linear algebra side there are vector spaces (cohomology theories) with certain relations and operations. In [62], Voevodsky gave an idea of the geometric side. He constructed the triangulated category of motives **DM**, equipped with a functor $M : Sm/k \to \mathbf{DM}$, where $Sm/k$ is the category of smooth $k$-schemes. The expected realisation functors from the geometric side to the linear algebra side were given by Huber([31]), Ayoub ([7, 8]), Ivorra ([32, 33]). The aim of the first part of this thesis is to extend the functor $M$ to the category of smooth separated Deligne-Mumford stacks.

Deligne-Mumford stacks are generalisation of schemes. They arise as solutions of moduli problems in algebraic geometry. A moduli problem is a functor $F : Sch^{op} \to Set$, where $Sch$ is the category of schemes. To give a solution to the moduli problem is to represent this $F$ by a scheme $X$. Often the moduli problem has no solution as the objects they classify have nontrivial automorphisms. After Deligne, Mumford and Artin ([53, 4]) a new tool, that of algebraic stacks became of great help in studying moduli problems. The main philosophy is that moduli problems do not really live in the category of functors $F : Sch^{op} \to Sets$ but rather in a 2-category of fibered categories over $Sch$. An algebraic stack, which arises as a solution to a moduli problem, not only classifies objects but also the isomorphisms of the objects. From this point of view algebraic stacks are more complicated objects than schemes, but it has the distinct advantage of retaining formally useful properties like smoothness when the underlying *coarse moduli space* would not. In this thesis, we work with the algebraic stacks introduced by Deligne-Mumford in [53]. Their definition is referred to as Deligne-Mumford stacks.

The study of Deligne-Mumford stacks from a motivic perspective began in [10] where the notion of the $DMC$-motive associated to a proper and smooth Deligne-Mumford stack was introduced as a tool for defining Gromov-Witten invariants. The construction of the category $\mathcal{M}_k^{DM}$ of $DMC$-motives uses $A^*$-Chow cohomology theories for Deligne-Mumford stacks as described in [24, 37, 36, 44, 60, 20]. These $A^*$-Chow cohomology theories coincide with rational coefficients. In [58, Theorem 2.1], Toen shows that the canonical functor $\mathcal{M}_k \to \mathcal{M}_k^{DM}$, from the category of usual Chow motives, is an equivalence rationally. In particular, to every smooth and proper Deligne-Mumford stack $M$, Toen associates a Chow motive $h(M)$.

In the first part of the thesis we construct motives for smooth (but not necessarily proper) Deligne-Mumford stacks as objects of Voevodsky's triangulated category of effective motives $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$. In the proper case, we compare these motives with the Chow motives we get using Toen's equivalence of categories $\mathcal{M}_k \simeq \mathcal{M}_k^{DM}$. Without assuming properness, our construction of the motive of a smooth Deligne-Mumford stack $F$ seems to be the first one.

The chapters are organised as follows.

In the first chapter, we briefly review Morel-Voevodsky $\mathbb{A}^1$-homotopy category $\mathbf{H}(k)$ ( resp. $\mathbf{H}^{\acute{e}t}(k)$) and Voevodsky's triangulated category of motives $\mathbf{DM}^{eff}(k,\mathbb{Q})$. We also construct the functor $M : \mathbf{H}^{\acute{e}t}(k) \to \mathbf{DM}^{eff}(k,\mathbb{Q})$. We recall those properties of Deligne-Mumford stacks which we use in the next chapters.

In the second chapter we study motives of Deligne-Mumford stacks. Given a presheaf of small groupoids $F$, we associate an object $Sp(F)$ in $\mathbf{H}^{\acute{e}t}(k)$. The motive of $F$ is defined to be $M(F) := M(Sp(F))$. We then show that for a Deligne-Mumford stack $F$ and an étale atlas $u : U \to F$ we have an isomorphism $M(U_\bullet) \cong M(F)$ in $\mathbf{DM}^{eff}(k,\mathbb{Q})$. Here $U_\bullet$ is the Čech hypercovering corresponding to the atlas $u : U \to F$. In section 2.2, we compare the motive of a separated Deligne-Mumford stack $F$ with the motive of the coarse moduli space of $F$. If $\pi : F \to X$ is the coarse moduli space of a separated Deligne-Mumford stack $F$, we show that the natural morphism $M(F) \to M(X)$ is an isomorphism in $\mathbf{DM}^{eff}(k,\mathbb{Q})$. We then prove projective bundle formula and blow-up formula for smooth Deligne-Mumford stacks. We also construct the Gysin triangle associated to a smooth, closed substack $Z$ of a smooth Deligne-Mumford stack $F$.

In section 2.3, we show that for any smooth and separated Deligne-Mumford stack $F$ over a field of characteristic zero, the motive $M(F)$ is a direct factor of the motive of a smooth quasi-projective scheme. If $F$ is proper we may take this scheme to be projective.

In section 2.4, we show that the motivic cohomology of a Deligne-Mumford stack (see [36, 3.0.2]) is representable in $\mathbf{DM}^{eff}(k,\mathbb{Q})$.

Finally in section 2.5 we compare our construction with Toen's construction and prove that for any smooth and proper Deligne-Mumford stack $F$ there is a canonical isomorphism $\iota \circ h(F) \cong M(F)$; this is Theorem II.27. Here $\iota : \mathcal{M}_k^{eff} \to \mathbf{DM}^{eff}(k,\mathbb{Q})$

is the fully faithful embedding described in [47, Proposition 20.1].

In the third chapter we study motivic decomposition of *relative geometrically cellular* Deligne-Mumford stacks. The results in this chapter are part of a joint project with Jonathan Skowera (see [16]). A *relative geometrically cellular* Deligne-Mumford stack is a smooth and separated Deligne-Mumford stack equipped with an increasing sequence of closed substacks whose successive differences, called *cells*, are affine fibrations over proper Deligne-Mumford stacks, called *bases*.

Even for varieties, our notion of relative cellularity is more general than the previous one given by Karpenko [38] (see also [27, Definition 3.1]: instead of asking that the fibers of the map from a cell to its base are affine spaces, we only ask so for the *geometric fibers*. By a result of Karpenko [38, Corollary 6.11], the Chow motive of a relative cellular variety decomposes into the direct sum of the Chow motives of the bases suitably shifted and twisted. We generalize this decomposition of Karpenko to include *geometrically* cellular Deligne-Mumford stacks (proposition III.13). In the classical case of relative cellular varieties, our method gives a simpler and more conceptual proof of Karpenko's decomposition theorem integrally. Our proof relies on a vanishing theorem of Voevodsky [62, Corollary 4.2.6] which says that there are no nonzero morphisms between motives of the form $M(X)$ and $M(Y)[1]$ for $X$ and $Y$ proper and smooth varieties.

In the last chapter, we study the homotopy invariance property of the connected component sheaves of spaces in the motivic homotopy theory setting. A functor $\mathcal{X} : \triangle^{op} \to PSh(Sm/k)$ is called a simplicial presheaf or a space. Here $\triangle$ is the category of simplices.

For any space $\mathcal{X}$, define $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ to be the presheaf

$$U \in Sm/k \mapsto Hom_{\mathbf{H}(k)}(U, \mathcal{X}).$$

The presheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant, i.e., for any $U \in Sm/k$ the morphism $\pi_0^{\mathbb{A}^1}(\mathcal{X})(U) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U)$, induced by the projection $\mathbb{A}^1_U \to U$, is bijective.

Let $a_{Nis} : PSh(Sm/k) \to Sh_{Nis}(Sm/k)$ denote the Nisnevich sheafification functor. The following conjecture of Morel states that the above property remains true after Nisnevich sheafification.

**Conjecture .1.** *For any $U \in Sm/k$, the morphism*

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_U),$$

*induced by the projection $\mathbb{A}^1_U \to U$, is bijective.*

We prove the conjecture (theorem IV.23) for $H$-groups (definition IV.9) and homogeneous spaces on these (see definitions IV.11, IV.12).

Voevodsky proved (see [47, Theorem 22.3]) that for any homotopy invariant presheaf with transfers $S$, the sheafification $a_{Nis}(S)$ is a homotopy invariant sheaf with transfers. The proof is quite hard. It becomes harder if we consider general homotopy invariant presheaves (without transfers). For any homotopy invariant presheaf of sets $S$ on $Sm/k$, one can ask to which extent the analogue of Voevodsky's result is true for $S$. Our results in this paper show that if $S$ is a presheaf of groups, the canonical morphism $S \to a_{Nis}(\pi_0^{\mathbb{A}^1}(S))$ is universal among all the morphisms from $S$ to homotopy invariant sheaves of sets.

The thesis ends with two appendixes. In Appendix 5.1, we show that some naturally defined functor $\omega : \mathcal{M}_k^{eff} \to PSh(\mathcal{V}_k)$ is fully faithful (see V.1). This statement appears without proof in [55, 2.2] and is mentioned in [58, page12]. It is also needed in the proof of II.27. In Appendix 5.2 we provide a technical result used in Appendix 5.1.

# TABLE OF CONTENTS

# CHAPTER I

# Introduction

In this chapter we review the results that we will use later. In the first section, we briefly review the constructions in motivic homotopy theory and Voevodsky's triangulated category of motives. In the second section we review some properties of Deligne-Mumford stacks.

## 1.1 Motivic homotopy theory and Voevodsky's triangulated category of motives

Recall that $\triangle$ is the category of simplices. The objects of this category are finite ordinals thought of as totally ordered sets, the morphisms are order preserving functions. The category of simplicial sets, denoted by $\triangle^{op}Sets$, is the category of functors $F : \triangle^{op} \to Sets$. Let $Sm/k$ be the category of smooth separated finite type $k$-schemes and denote $PSh(Sm/k)$ the category of presheaves of sets on $Sm/k$. Also denote $\triangle^{op}PSh(Sm/k)$ the category of spaces, i.e., presheaves of simplicial sets. We denote by $\mathbf{H}_s^{\acute{e}t}(k)$ (resp. $\mathbf{H}_s(k)$) the homotopy category of $\triangle^{op}PSh(Sm/k)$ with respect to the étale (resp. Nisnevich) local model structure, i.e., obtained by inverting formally the étale (resp. Nisnevich) local weak equivalences. Following [48, §3.2], we consider the Bousfield localisation of the étale (resp. Nisnevich) local model structure on $\triangle^{op}PSh(Sm/k)$ with respect to the class of maps $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ for all spaces $\mathcal{X}$. The resulting model structure will be called the étale (resp. Nisnevich) motivic model structure. The homotopy category with respect to this étale (resp. Nisnevich) motivic model structure is denoted by $\mathbf{H}^{\acute{e}t}(k)$(resp $\mathbf{H}(k)$). (We warn the reader that in [48, §3.2] the Nisnevich topology is used instead of the étale topology.)

**Remark I.1.** *Denote* $(\triangle^{op}PSh(Sm/k))_\bullet$ *the category of pointed spaces, i.e., presheaves of pointed simplicial sets on* $Sm/k$*. We also have the pointed versions of the local and motivic model structures where weak equivalences are detected after forgetting the base point. The pointed étale (resp. Nisnevich) homotopy category is denoted by* $\mathbf{H}^{\acute{e}t}_{s,\bullet}(k)$ *(resp.* $\mathbf{H}_{s,\bullet}(k)$*). The pointed étale (resp. Nisnevich) motivic homotopy categoriy is denoted by* $\mathbf{H}^{\acute{e}t}_\bullet(k)$ *(resp.* $\mathbf{H}_\bullet(k)$*).*

For the projective global model structure on $\triangle^{op}PSh(Sm/k)$, the weak equivalences (resp. fibrations) are the sectionwise weak equivalences (resp. fibrations). Cofibrations are defined using the left lifting property with respect to the trivial fibrations. The left Bousfield localisation with respect to the étale (resp. Nisnevich) hypercovers gives the étale (resp. Nisnevich) local projective model structure. We can further localise the étale (resp. Nisnevich) local projective model structures with respect to the class of maps $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$. The resulting homotopy category is equivalent to $\mathbf{H}^{\acute{e}t}(k)$ (resp. $\mathbf{H}(k)$).

Let $Grpd$ be the category of small groupoids and let $Grpd(Sm/k)$ be the category of presheaves of small groupoids on $Sm/k$.

For any $X \in \triangle^{op}Sets$, the fundamental groupoid $\pi(X)$ is defined to be the free groupoid on the path category $P(X)$. $Ob(P(X)) = X_0$, and $P(X)$ is generated as a category by the 1-simplices $x \to y$ of $X$ subject to the relation $d_1(\sigma) = d_0(\sigma)d_2(\sigma)$ for each 2-simplex $\sigma$ of $X$. This gives a functor $\pi : \triangle^{op}PSh(Sm/k) \to Grpd(Sm/k)$.

The functor $\pi : \triangle^{op}Sets \to Grpd$ has a right adjoint, $Ner : Grpd \to \triangle^{op}Sets$, given by for any simplicial degree $k$ and any groupoid $G$,

$$Ner(G)_k := Hom_{Grpd}(\pi(\triangle^k), G).$$

Hence $Ner : Grpd(Sm/k) \to \triangle^{op}PSh(Sm/k)$ is right adjoint to

$$\pi : \triangle^{op}PSh(Sm/k) \to Grpd(Sm/k).$$

For the étale (resp. Nisnevich) local model structure on $Grpd(Sm/k)$, a morphism $f : G \to H$ in $Grpd(Sm/k)$ is an étale (resp. Nisnevich) local weak equivalence (resp. fibration) if $Ner(f) : Ner(G) \to Ner(H)$ is an étale (resp. Nisnevich) local weak equivalence (resp. fibration). Cofibrations are the morphisms which has the left lifting property with respect to the trivial fibrations.

Recall that $SmCor(k)$ is the category of finite correspondences. Objects of this category are smooth $k$-schemes $X$. For $X$, $Y \in Sm/k$, $Hom_{SmCor(k)}(X, Y)$ is given by the group of finite correspondences $Cor(X, Y)$. This is the free abelian group generated by integral closed subschemes $W \subset X \times Y$ which are finite and surjective on a connected component of $X$. Thus, if $X = \coprod_i X_i$ we have

$$Cor(X \times Y) = \bigoplus_i Cor(X_i \times Y)$$

.

A presheaf with transfers is a contravariant additive functor on $SmCor(k)$. Denote by $PST(k, \mathbb{Q})$ the category of presheaves with transfers with values in the category of $\mathbb{Q}$-vector spaces. A typical example is given by $\mathbb{Q}_{tr}(X)$ for $X \in Sm/k$. This presheaf associates to each $U \in Sm/k$ the vector space $Cor(U, X) \otimes \mathbb{Q}$.

Analogous to the local and motivic model category structures on the category of spaces, we have a local and a motivic model structures on the category $K(PST(k, \mathbb{Q}))$

of complexes of presheaves with transfers. A morphism $K \to L$ in $K(PST(k,\mathbb{Q}))$ is an étale weak equivalence if it induces quasi-isomorphisms on stalks for the étale topology (or the Nisnevich topology; it doesn't matter in the presence of transfers). Cofibrations are monomorphisms and fibrations are characterized by the right lifting property. This gives the local model category structure on $K(PST(k,\mathbb{Q}))$ (cf. [5, Theorem. 2.5.7]). The homotopy category is nothing but the derived category of étale sheaves with transfers $\mathbf{D}(Str(Sm/k,\mathbb{Q}))$. (Here $Str(Sm/k,\mathbb{Q})$ is the category of étale sheaves with transfers).

The motivic model structure is the Bousfield localisation of the local model structure with respect to the class of morphisms $\mathbb{Q}_{tr}(\mathbb{A}^1 \times X)[n] \to \mathbb{Q}_{tr}(X)[n]$ for $X \in Sm/k$ and $n \in \mathbb{Z}$. The resulting homotopy category with respect to the motivic model structure is denoted by $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$. This is Voevodsky's triangulated category of mixed motives (with rational coefficients).

The functor $\mathbb{Q}_{tr}(-) : Sm/k \to PST(k,\mathbb{Q})$ extends to a functor

$$\mathbb{Q}_{tr} : PSh(Sm/k) \to PST(k,\mathbb{Q})$$

given by $\mathbb{Q}_{tr}(F) = Colim_{X \to F}\, \mathbb{Q}_{tr}(X)$ for any presheaf of sets $F$ on $Sm/k$.

Let $A$ be a simplicial abelian group. The normalized chain complex associated to $A$ is the complex $NA$, such that for all $n \in \mathbb{N}$, $NA_n := \bigcap_{i=0}^{n-1} kerd_i \subset A_n$, and the differential is given by $\delta = \sum_{i=0}^{n}(-1)^i d_i$.

In the next statement, $N(-)$ denotes the functor that associates the presheaf of normalized chain complex to a simplicial abelian presheaf (cf. [25, page 145]).

**Proposition I.2.** *There exists a functor $M : \mathbf{H}^{\acute{e}t}(k) \to \mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$. It sends a simplicial scheme $X_\bullet$ to $N\mathbb{Q}_{tr}(X_\bullet)$.*

*Proof.* This is well known. We give a sketch of proof here. The functor $\mathbb{Q}_{tr}(-)$ : $PSh(Sm/k) \to PST(k, \mathbb{Q})$ extends to a functor

$$N\mathbb{Q}_{tr}(-) : \triangle^{op} PSh(Sm/k) \to K(PST(k, \mathbb{Q})).$$

There is a functor $\Gamma : K(PST(k, \mathbb{Q})) \to \triangle^{op} PSh(Sm/k)$ right adjoint to $N\mathbb{Q}_{tr}$ (cf. [25, page 149]). We will show that the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen adjunction for the projective motivic model structures. (These model structures are different from the ones described above: they have the same weak equivalences but the cofibrations in the projective ones are defined by the left lifting property with respect to section-wise trivial fibrations of presheaves of simplicial sets and surjective morphisms of complexes of presheaves with transfers respectively.)

The functor $\Gamma$ takes section-wise weak equivalences in $K(PST(k, \mathbb{Q}))$ to section-wise weak equivalences in $\triangle^{op} PSh(Sm/k)$ and it takes surjective morphisms to section-wise fibrations in $\triangle^{op} PSh(Sm/k)$. Hence for the projective global model structures the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen adjunction. By [19, Theorem 6.2] the projective étale local model structure on $\triangle^{op} PSh(Sm/k)$ is the Bousfield localisation of the global projective model structure with respect to general hypercovers for the étale topology. Let $S$ be the class of those hypercovers. To show that the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen adjunction for the étale local model structures, we need to show that the left derived functor of $N\mathbb{Q}_{tr}$ maps morphisms in $S$ to étale local weak equivalences in $K(PST(k, \mathbb{Q}))$. For this it is enough to show that $\Gamma$ maps a local fibrant object $C_\bullet$ of $K(PST(k, \mathbb{Q}))$ to an $S$-local object of $\triangle^{op} PSh(Sm/k)$. Showing that $\Gamma(C_\bullet)$ is $S$-local is equivalent to showing that the étale hypercohomology $\mathbb{H}^n_{ét}(X, C_\bullet)$ is isomorphic to $H^n(C_\bullet(X))$ for any $X \in Sm/k$ and $n \geq 0$. Now, the hypercohomology $\mathbb{H}^n_{ét}(X, C_\bullet)$ can be calculated using Čech hypercovers $U_\bullet \to X$ and moreover by [47, Prop 6.12] the complex $\mathbb{Q}_{tr}(U_\bullet)$ is a resolution of the étale sheaf $\mathbb{Q}_{tr}(X)$. Since $C_\bullet$ is local fibrant we have

$$H^n(C_\bullet(X)) = Hom_{Ho(K(PST(k,\mathbb{Q})))}(\mathbb{Q}_{tr}(U_\bullet), C_\bullet[n]) = H^n(Tot(C_\bullet(U_\bullet))).$$

(Here $Ho(K(PST(k,\mathbb{Q})))$ is the homotopy category with respect to global projective model structure on $K(PST(k,\mathbb{Q}))$.) Now passing to the colimit over hypercovers $U_\bullet \to X$ we get $\mathbb{H}^n_{ét}(X, C_\bullet) \cong H^n(C_\bullet(X))$.

At this point we get a functor $\mathbf{H}^{ét}_s(k) \to \mathbf{D}(Str(Sm/k))$ and it remains to show that this functor takes the maps $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ to motivic weak equivalences. This is clear by construction. $\square$

**Remark I.3.** *There is also a functor $M : \mathbf{H}^{ét}(k) \to \mathbf{DM}^{\mathit{eff},\, ét}(k,\mathbb{Z})$ to Voevodsky's category of étale motives with integral coefficients. It is constructed exactly as above. (Note that with integral coefficients, the categories of étale and Nisnevich motives are different: we denote them $\mathbf{DM}^{\mathit{eff},\, ét}(k,\mathbb{Z})$ and $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})$ respectively; with rational coefficient these categories are the same.)*

Let $\triangle^\bullet$ be the cosimplicial scheme defined by :

$$\triangle^n = Spec(k[x_0, \dots, x_n]/(\sum_{i=0}^{n} x_i = 1).$$

The $j$-th face map $\delta_j$ is given by the equation $x_j = 0$. Given a presheaf of abelian groups $F$ on $Sm/k$, we can construct the simplicial abelian group $C_\bullet F$ such that for any $U \in Sm/k$, $C_\bullet F(U)_n = F(U \times \triangle^n)$. Let $C_*F := N(C_\bullet F)$ be the associated normalized complex of presheaf with transfers.

**Definition I.4.** *For every integer $q \geq 0$ and any abelian group $A$ the motivic complex $A(q)$ is defined as the following complex of presheaves with transfers :*

$$A(q) = C_* A_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

**Definition I.5.** *For any abelian group $A$, the étale (or Lichtenbaum) motivic cohomology of $X$ is defined as the étale hypercohomology of $A(q)$:*

$$H_L^{p,q}(X,A) = \mathbb{H}^p_{et}(X, A(q)|_{X_{et}})$$

*and the motivic cohomology groups $H^{p,q}(X,A)$ is defined as the Zariski hypercohomology*

$$H^{p,q}(X,A) = \mathbb{H}^p_{Zar}(X, A(q)).$$

The following properties of $\mathbf{DM}^{eff}(k,\mathbb{Q})$ will be useful in the next chapters.

**Properties I.6.** *1. If $E \to X$ is a vector bundle, then the induced morphism $M(E) \to M(X)$ is an isomorphism. If $\mathbb{P}(\mathcal{E}) \to X$ is a projective bundle of rank $n+1$, then*

$$\oplus_{i=0}^n M(X)(i)[2i] \cong M(\mathbb{P}(\mathcal{E})).$$

*2. Assume that resolution of singularities holds over $k$. Let $X' \to X$ be a blow-up with center $Z$, and set $Z' = Z \times_X X'$. There is a distinguished triangle :*

$$M(Z') \to M(X') \oplus M(Z) \to M(X) \to M(Z')[1].$$

*Moreover if $X$ and $Z$ are smooth , and $Z$ has codimension c, then*

$$M(X') \cong M(X) \oplus (\oplus_{i=1}^{c-1} M(Z)(i)[2i]).$$

*3. Let $X$ be a smooth scheme over a perfect field and let $Z$ be a smooth closed subscheme of $X$ of codimension c. Then there is a Gysin triangle :*

$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].$$

*4. For any abelian group $A$, $H^{2i,i}(X,A) = CH^i(X) \otimes A$. If $A$ is a $\mathbb{Q}$-vector space, then*

$$H^{n,i}(X,A) \cong H_L^{n,i}(X,A) \cong Hom_{\mathbf{DM}^{eff}(k,\mathbb{Q})}(M(X), A(i)[n]).$$

5. *There exists a fully faithful embedding* $\iota : \mathcal{M}_k^{eff} \to \mathbf{DM}^{eff}(k, \mathbb{Q})$*, such that* $\iota(h(X)) \cong M(X)$ *for smooth and projective* $X$*. Here,* $\mathcal{M}_k^{eff}$ *is the category of effective Chow motives with rational coefficients and* $h(X)$ *is the Chow motive of the smooth projective variety* $X$*.*

### 1.2 Deligne-Mumford stacks

Let $\mathcal{C}$ be any category. Consider $2-Fun(\mathcal{C}^{op}, Grpd)$ the category of lax 2-functors from $\mathcal{C}$ to *Grpd*. Recall that a lax 2-functor $F$ associates to $X \in \mathcal{C}$ a groupoid $F(X)$, to $f : Y \to X$ a functor $F(f) : F(X) \to F(Y)$, and to composable morphisms $f$ and $g$ an isomorphism $F(f) \circ F(g) \cong F(g \circ f)$. The 1-morphisms between two lax 2-functors $\mathcal{F}$ and $\mathcal{G}$ are lax natural transformations $H$ such that for any $f : Y \to X \in \mathcal{C}$ there is a natural isomorphism between the fucntors $G(f) \circ H_X$ and $H_Y \circ F(f)$. For any composable morphisms $f$ and $g$, we have the usual compatibility conditions. 2-isomorphisms between lax transformations $H$ and $H'$ are given by isomorphisms of functors $a_X : H_X \cong H'_X$ for each $X \in \mathcal{C}$, such that for any $f : Y \to X$ we have $G(f)(a_X) = a_Y(F(f))$.

For objects $X$, $Y \in \mathcal{C}$, consider the set $Hom_{\mathcal{C}}(Y, X)$ as a discrete groupoid, i.e, all morphisms are identities. In this way, the functor $Hom_{\mathcal{C}}(-, X) : \mathcal{C} \to Grpd$ is a strict 2-functor which we denote by $h(X)$.

**Lemma I.7.** *Let* $F \in 2-Fun(\mathcal{C}^{op}, Grpd)$*. There is a surjective equivalence of categories* $Hom_{2-Fun(\mathcal{C}^{op},Grpd)}(h(X), F) \to F(X)$ *given by evaluating at* $id_X \in h(X)(X)$*.*

*Proof.* Given any lax natural transformation $H : h(X) \to F$ we get an object $X' := H_X(id_X) \in F(X)$. Given two lax natural transformations $H, H'$ and a 2-isomorphism $a$ between them, we get an isomorphism $a_X(id_X) : H_X(id_X) \cong H'_X(id_X)$. Let $X' \in F(X)$. We have a natural transformation given by $G_Y(f : Y \to X) = F(f)(X')$.

Since $F$ is a lax presheaf we have $F(f \circ g)(X') \cong F(g) \circ F(f)$ for any $Z \xrightarrow{g} Y \xrightarrow{f} X$. Hence we get the required natural transformation between $F(g) \circ G_Y$ and $G_Z \circ h(X)(g)$. Moreover let $H, G : h(X) \to F$ such that there exists a morphism $f : H_X(id_X) \to G_X(id_X) \in F(X)$. We define a unique 2-isomorphism $a$ between $H$ and $G$ in the following way. For any $g \in h(X)(Y)$ we have $H_Y(g) \cong F(g)(H_X(id_X))$ given by the structure of the lax natural transformation. Similarly we get $G_Y(g) \cong F(g)(G_X(id_X))$. But then there exists $F(g)(f) : F(g)(H_X(id_X)) \cong F(g)(G_X(id_X))$. So $a_Y(g) : H_Y(g) \cong G_Y(g)$ and $a_X(id_X) : H_X(id_X) \to G_X(id_X)$ is equal to $f$. $\square$

**Remark I.8.** *In general $Hom_{2-Fun(\mathcal{C}^{op},Grpd)}(h(X), F)$ is not small unless $\mathcal{C}$ is small. Let $Sch/k$ be the category of finite type $k$-schemes. We fix $C \subset Sch/k$ which is a full small subcategory equivalent to $Sch/k$. For any $X \in Sch/k$ and $F \in 2-Fun((Sch/k)^{op}, Grpd)$ the association $X \mapsto Hom_{2-Fun(C^{op},Grpd)}(h(X)|_C, F|_C)$ gives a strict presheaf of groupoids. We denote it by $h_{st}(F)$ . By I.7 we have an equivalence $F|_C \cong h_{st}(F)|_C$.*

Let $F : (Sch/k)^{op} \to Grpd$ be a lax 2-functor.

**Definition I.9.** *The functor $F$ is a stack in the étale topology if it satisfies the following axioms (descent) where $\{f_i : U_i \to U\}_{i \in I}$ is an étale covering of $U \in Sch/k$ and $f_{ij,i} : U_i \times_U U_j \to U_i$ are the projections.*

1. *(Glueing of morphisms) If $X$ and $Y$ are two objects of $F(U)$, and $\phi_i : F(f_i)(X) \cong F(f_i)(Y)$ are isomorphisms such that $F(f_{ij,i})(\phi_i) = F(f_{ij,j})(\phi_j)$, then there exists an isomorphism $\eta : X \cong Y$ such that $F(f_i)(\eta) = \phi_i$.*

2. *(Separation of morphisms) If $X$ and $Y$ are two objects of $F(U)$, and $\phi : X \cong Y$, $\psi : X \cong Y$ are isomorphisms such that $F(f_i)(\phi) = F(f_i)(\psi)$, then $\phi = \psi$.*

3. *(Glueing of objects) If $X_i$ are objects of $F(U_i)$ and $\phi_{ij} : F(f_{ij,j})(X_j) \cong F(f_{ij,i})(X_i)$ are isomorphisms satisfying the cocycle condition*

$$(F(f_{ijk,ij})(\phi_{ij})) \circ (F(f_{ijk,jk})(\phi_{jk})) = F(f_{ijk,ik})(\phi_{ik}),$$

*then there exist an object $X$ of $F(U)$ and $\phi_i : F(f_i)(X) \cong X_i$ such that $\phi_{ji} \circ (F(f_{ij,i})(\phi_i)) = F(f_{ij,j})(\phi_j)$.*

**Remark I.10.** *1. Any presheaf of groupoids that satisfies descent with respect to every Čech cover satisfies descent with respect to every hypercover [19, corollary A.9]. Let $F : (Sch/k)^{op} \to Grpd$ be a lax 2-functor. The strict presheaf of groupoids $h_{st}(F)$ is a stack if and only if $F$ is a stack by [29]. So $F$ is a stack if and only if $Ner(h_{st}(F))$ is fibrant in the étale local projective model structure.*

*2. There is a notion of strict stacks (see [29], [35]). If $F$ is a strict presheaf of groupoids then by [35, lemma 7, lemma 9] there exists a strict stack $St(F)$ and a morphism $st : F \to St(F)$ such that $st$ is a local weak equivalence, i.e., $st$ induces equivalences of groupoids on stalks. The stack $St(F)$ is called the associated stack of $F$ and the functor is called the stackification functor.*

**Definition I.11.** *Let $F, G, H \in Grpd(Sch/k)$, such that there exists morphisms $f : G \to F$ and $g : H \to F$. The homotopy fiber product $G \times_F H$ is defined as follows : For any $X \in Sm/k$, the objects of $G \times_F H$ are tuples $(x_1, x_2, a)$, where $x_1 \in Ob(G(X))$, $x_2 \in Ob(H(X))$, and $a : f(x_1) \to g(x_2)$. A morphism between $(x_1, x_2, a) \to (y_1, y_2, b)$ is a tuple $(\alpha, \beta)$ such that $\alpha : x_1 \to y_1$ and $\beta : x_2 \to y_2$, and we have the following commutative digram :*

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\alpha)"] \arrow[d, "a"] & f(y_1) \arrow[d, "b"] \\ g(x_2) \arrow[r, "g(\beta)"] & g(y_2) \end{tikzcd}$$

**Remark I.12.** *If $F, G, H$ are presheaves of discrete groupoids, i.e., presheaves of sets, then the homotopy fiber product is the usual fiber product.*

**Definition I.13.** *Let $F, G \in Grpd(Sch/k)$. A morphism $f : F \to G$ is called representable by a scheme, if for any $g : h(X) \to G$, where $X$ is a $k$-scheme, the homotopy fiber product $h(X) \times_G F \cong h(Y)$ for some scheme $Y$.*

Let $P$ be a property of morphisms $f : X \to Y$ of schemes, stable by base change and local for the étale topology on $Y$. For example, surjective, separated, quasicompact, open(closed) immersions, affine (quasi affine), finite (quasi finite), proper, flat, unramified, smooth, étale morphisms satisfies this property.

**Definition I.14.** *A representable morphism $f : F \to G$ of stacks has property $P$, if for any scheme $U$ and a morphism $u : h(U) \to G$, the canonical morphism of schemes $h(U) \times_G F \to h(U)$ has property $P$.*

For any groupoid object $R \rightrightarrows U$ in $Sch/k$ we can associate a strict presheaf of groupoids $h(R \rightrightarrows U)$ (see proof of lemma I.24).

**Definition I.15.** *A Deligne-Mumford stack $F$ is a stack on $Sch/k$ admitting a local equivalence (stalk-wise equivalence in the étale topology) $h(R \rightrightarrows U) \to F$, where $R \rightrightarrows U$ is a groupoid object in $Sch/k$, such that both morphisms $R \to U$ are étale and $R \to U \times_k U$ is finite.*

**Remark I.16.** *Recall (see [46, 53]), that a separated finite type Deligne-Mumford stack is a stack $F$, such that the diagonal $\triangle : F \to F \times_k F$ is representable and finite and there exists an étale surjective morphism $a : U \to F$. It is clear that, $U \times_F U \rightrightarrows U$ is a groupoid object in $Sch/k$, $h(U \times_F U \rightrightarrows U) \to F$ is an étale weak equivalence and both projections $U \times_F U \to U$ are étale and $U \times_F U \to U \times_k U$ is finite.*

*Conversely, let* $R \rightrightarrows U$ *be a groupoid object in* $Sch/k$*, such that both morphisms* $R \to U$ *are étale and* $R \to U \times_k U$ *is finite. Recall that there exists a separated Deligne-Mumford stack* $F$ *and an étale local weak equivalence* $h(R \rightrightarrows U) \to F$ *such that* $U \to F$ *is étale and surjective and* $R \cong U \times_F U$*. If* $F'$ *is any other stack equipped with an étale local weak equivalence* $h(R \rightrightarrows U) \to F$*, then* $F \cong F'$*. Hence, our definition of a Deligne-Mumford stack is equivalent to that of separated finite type Deligne-Mumford stack from [46, 53]. The morphism* $p : U \to F$ *is called an atlas of* $F$*.* $F$ *is smooth if and only if* $U$ *is smooth.*

**Definition I.17.** *Let* $F$ *be a Deligne-Mumford stack. A coarse moduli space for* $F$ *is a map* $\pi : F \to X$ *to an algebraic space* $X$ *such that* $\pi$ *is initial among maps from* $F$ *to algebraic spaces, and for every algebraically closed field* $k$ *the map* $[F(k)] \to X(k)$ *is bijective (where* $[F(k)]$ *denotes the set of isomorphism classes of objects in the small category* $F(k)$*).*

**Theorem I.18.** *For any separated finite type Deligne-Mumford stack* $F$*. There exists a coarse moduli space* $\pi : F \to X$*. The morphism* $\pi$ *is proper and an universal homeomorphism, and the space* $X$ *is separated. Moreover, if* $char(k) = 0$ *and* $Y \to X$ *any morphism of schemes, then* $Y$ *is the moduli space of the Deligne-Mumford stack* $F \times_X Y$*.*

*Proof.* [39, 18]. $\square$

**Remark I.19.** *Since the morphism* $\pi : F \to X$ *is a proper universal homeomorphism,* $F$ *is proper if and only if* $X$ *is proper.*

**Example I.20.** *Let* $X$ *be a finite type* $k$*-scheme and let* $G$ *be a smooth affine group scheme acting on* $X$ *such that the geometric stabilizers of the geometric points are finite and reduced. The quotient lax-presheaf of groupoids* $[X/G]$ *is defined as follows.*

*For any scheme $Y$, the objects of $[X/G](Y)$ are $G$-principal bundles $E \to Y$ together with a $G$-equivariant map $f : E \to X$. Morphisms from $(E \to Y, E \to X)$ to $(E' \to Y, E' \to X)$ is given by $G$-equivariant isomorphism $g : E \to E'$. For any $h : Y' \to Y$, the restriction morphism is given by fixing some pullback of a $G$-bundle over $Y$ to $Y'$. This is a Deligne-Mumford stack. It is separated if and only if the action is proper.*

*If $G$ is finite and $X$ is separated, then the geometric quotient $X/G$ exists in the category of algebraic spaces. By [43, corollary 2.15] $X/G$ is also the categorical quotient in the category of algebaric spaces. Hence $X/G$ is the coarse moduli space of $[X/G]$ (this can be extended to $G$ affine reductive group acting properly on a scheme $X$).*

**Theorem I.21.** *Let $F$ be a Deligne-Mumford stack, and let $p : F \to X$ be its coarse moduli space. Then, there exists an étale covering $(U_i \to X)_{i\in I}$ of $X$, and finite groups $G_i$, and quasi-projective schemes $X_i$ with an action of $G_i$, such that for all $i \in I$, the Deligne-Mumford stacks $F \times_X U_i$ is equivalent to the quotient stack $[X_i/G_i]$.*

*Proof.* [60, 2.8] [59, Prop 1.17] □

**Remark I.22.** *Let $F$ be a Deligne-Mumford stack, and let $X$ be its coarse moduli space. Suppose $([X_i/G_i] \to F)_{i\in I}$ be the étale covering of $F$ deduced from the covering of moduli space $(X_i/H_i \to X)_{i\in I}$ as in the previous theorem. Then $[X_i/G_i] \times_F [X_j/G_j] \cong [(X_j \times_X (X_i/H_i))/H_j]$, and hence a quotient stack.*

**Remark I.23.** *Let $F, F'$ be Deligne-Mumford stacks. Let $U \to F$ be an atlas of $F$. Let $s$ and $t$ be the morphisms $U \times_F U \to U$ associated to the groupoid object $(U \times_F U \rightrightarrows U)$. Suppose $f : U \to F'$ is given such that $s \circ f \cong t \circ f$ in $F'(U \times_F U)$ which satisfies cocycle condition in $F'(U \times_F U \times_F U)$. Then we get a morphism*

*$f : F \to F'$ as follows. For any $x \in F(T)$ we get a map of local sections $U \times_F T \to T$ and an element $F'(U \times_F T)$ together with a glueing data on $U \times_F U \times_F T$ and cocycle condition. This defines an element in $F'(T)$. More generally for any class of objects which satisfy descent, i.e., which can be defined étale locally by glueing data, we can define the corresponding objects over stacks by glueing-data on one atlas.*

*For example, let $F$ be a separated Deligne-Mumford stack, and let $U \to F$ be an atlas. Let $Z$ be a closed substack of $F$. The blow-up of $F$ centerd at $Z$ is defined as follows. Let $R := U \times_F U$ and let $(R \rightrightarrows U)$ be the associated groupoid scheme. Let $Z_U$ and $Z_R$ be the pullback of $Z$ to $U$ and $R$ respectively. Using universal property of blow-ups and the fact that blow-ups commutes with smooth base change, we get an étale groupoid scheme $(Bl_{Z_R}(R) \rightrightarrows Bl_{Z_U}(U))$, and morphism of groupoid objects $(Bl_{Z_R}(R) \rightrightarrows Bl_{Z_U}(U)) \to (R \rightrightarrows U)$. We get a separated Deligne-Mumford stack $Bl_Z(F)$, which is the stackification of $(Bl_{Z_R}(R) \rightrightarrows Bl_{Z_U}(U))$. To construct the morphism $f : Bl_Z(F) \to F$, we follow the description given in the first paragraph. The morphism $f$ is representable and projective. For any atlas $U \to F$, the pullback $U \times_F U \cong Bl_{Z_U}(U)$. One can similarly give constructions of vector bundles and porjective bundles.*

Given an atlas $f : U \to F$ of a smooth Deligne-Mumford stack $F$, we get a simplicial object $U_\bullet$ in $Sm/k$ by defining $U_i = U \times_F \cdots \times_F U$ $(i+1$ times) and the face and degeneracy maps are defined by relative diagonal and partial projections.

**Lemma I.24.** *For $R \rightrightarrows U$ as above we have $Ner(h(R \rightrightarrows U)) = U_\bullet$.*

*Proof.* By definition we have $Ob(h(R \rightrightarrows U)(S)) = Hom_{Sm/k}(S, U)$ and $Mor(h(R \rightrightarrows U)(S)) = Hom_{Sm/k}(S, R)$. The set of two composable morphisms in $h(R \rightrightarrows U)(S)$ is $(R \times_U R)(S)$ where $R \times_U R$ is the fiber product of the maps $s : R \to U$ and

$t : R \to U$. More generally the set of $n$-composable morphisms in $h(R \rightrightarrows U)(S)$ is $R \times_U R \times_U \cdots \times_U R$ ($n$ times). Since $R \cong U \times_F U$ and the maps $s$ and $t$ are first and second projections respectively we have

$$(Ner(h(R \rightrightarrows U)))_n(S) = \underbrace{(U \times_F U) \times_U \cdots \times_U (U \times_F U)}_{n \text{ times}}$$

which is isomorphic to $U_n$. □

**Theorem I.25.** *Let $U \to F$ be an atlas for a smooth Deligne-Mumford stack $F$. There is a canonical étale local weak equivalence $U_\bullet \to Sp(F)$.*

*Proof.* We know that $h(R \rightrightarrows U)$ is locally weakly equivalent to $F$. Hence the morphism $Ner(h(R \rightrightarrows U)) \to Sp(F)$ is a local weak equivalence. The claim follows now from lemma I.24. □

**Definition I.26.** *The étale site $F_{ét}$ is defined as follows. The objects of $F_{ét}$ are couples $(X, f)$ with $X$ a scheme and $f : X \to F$ a representable étale morphism. A morphism from $(X, f)$ to $(Y, g)$ is a couple $(\phi, \alpha)$, where $\phi : X \to Y$ is a morphism of schemes and $\alpha : f \cong g \circ \phi$ is a 2-isomorphism. Covering families of an object $(U, u)$ are defined as families $\{u_i : U_i \to U\}_{i \in I}$ such that the $u_i$'s are étale and $\cup u_i : \coprod_i U_i \to U$ is surjective.*

For each integer $i$ let $\mathbb{Q}(i) \in \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$ denote the motivic complex of weight $i$ with rational coefficients (see definition I.4).

**Definition I.27.** *Let $F$ be a Deligne-Mumford stack. The motivic cohomology of $F$ with rational coefficients is defined as $H_M^{2i-n}(F, i) := H^{2i-n}(F_{ét}, \mathbb{Q}(i)|_{F_{ét}})$.*

**Remark I.28.** *In [36, 3.0.2], motivic cohomology of an algebraic stack $F$ is defined using the smooth site of $F$. For Deligne-Mumford stacks, this coincides with our definition by [36, Proposition 3.6.1(ii)].*

For a Deligne-Mumford stack $F$, consider the sheaf of the $m$-th $K$-groups $\mathcal{K}_m$ on $F_{et}$ associated with the presheaf

$$K_m : F_{et} \to Ab$$

$$U \mapsto K_m(U).$$

**Definition I.29.** *The codimension $m$-rational Chow group of $F$ is defined as*

$$A^m(F) := H^m(F_{et}, \mathcal{K}_m \otimes \mathbb{Q})$$

.

**Remark I.30.** *By [36, Theorem 3.1, 5.3.10], we have a functorial isomorphism*

$$A^m(F) \cong H^{2m}_M(F, m)_{\mathbb{Q}}.$$

Recall the construction of the category of Chow motives for smooth and proper Deligne-Mumford stacks using the theory $A^*$ ([10, §8]). For proper smooth Deligne-Mumford stacks $F$ and $F'$, the vector space of correspondences of degree $m$ between $F$ and $F'$ is defined to be

$$S^m(F, F') := \left\{x \in A^*(F \times F')/(p_2)_*(x) \in A^m(F')\right\},$$

where $p_2 : F \times F' \to F'$ is the second projection. We have the usual composition

$$\circ : S^m(F, F') \otimes S^n(F', F'') \to S^{p+m}(F, F'')$$

given by the formula

$$x \circ y := (pr_{13})_*(pr_{12}^*(x).pr_{23}^*(y)),$$

where the $pr_{ij}$ are the natural projections of $F \times F' \times F''$ on two of the three factors. Objects of $\mathcal{M}_k^{DM}$ are triples $(F, p, m)$, such that $F$ is a smooth proper Deligne-Mumford stack, $p$ is an idempotent in the ring of correspondences $S^0(F, F)$, and

$m \in \mathbb{Z}$. The vector space of morphisms between $(F, p, m)$ and $(F', q, n)$ is given by

$$Hom_{\mathcal{M}_k^{DM}}((F, p, m), (F', q, n)) := q \circ S^{n-m}(F, F') \circ p \subset S^{n-m}(F, F').$$

The full subcategory of $\mathcal{M}_k^{DM}$ consisting of objects $(F, p, 0)$ is called the category of effective Chow motives and we denote it by $\mathcal{M}_k^{DM,\,\mathit{eff}}$. The category $\mathcal{M}_k^{DM}$ is a $\mathbb{Q}$-linear symmetric monoidal karoubian category.

**Theorem I.31.** *There is a natural fully faithful tensorial functor*

$$\mathcal{M}_k \to \mathcal{M}_k^{DM},$$

*which is an equivalence of* $\mathbb{Q}$*-tensorial category.*

*Proof.* [58, theorem 2.1]. □

# CHAPTER II

## Motives of Deligne-Mumford Stacks

In this chapter we first define the motive of any lax presheaf of groupoids on $Sm/k$ (II.1). By lemma II.2 we get a resolution of $M(F)$ by the motive of any Čech cover, where $F$ is a smooth Deligne-Mumford stack. The isomoprhism between motive of a smooth Deligne-Mumford stack and motive of its coarse moduli space is given in theorem II.6. The main formulas, which are useful for computations, are theorem II.7, proposition II.10 and theorem II.13. We prove in theorem II.20 that motive of a smooth separated Deligne-Mumford stack is a direct factor of a smooth quasi-projective variety. We end this chapter with the comparison theorem II.27. The results in this chapter are part of [15].

### 2.1 The general construction

In this section we describe our construction of the motive associated to a smooth Deligne-Mumford stack. In fact, our construction applies more generally to any stack but the existence of atlases can be used to give explicit models.

**Definition II.1.** *Let $F \in 2{-}Fun((Sch/k)^{op}, Grpd)$. Then the motive of $F$ is defined as $M(F) := M(Sp(F))$. This gives a functor*

$$M : 2 - Fun((Sch/k)^{op}, Grpd) \to \mathbf{DM}^{eff}(k, \mathbb{Q}).$$

*Using I.3 we can also define an integral version of the motive of $F$, which we also denote $M(F)$ if no confusion can arise.*

**Lemma II.2.** *Let $U \to F$ be an atlas for a smooth Deligne-Mumford stack $F$. The canonical map $M(U_\bullet) \to M(F)$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$ is an isomorphism. (This is also true integrally.)*

*Proof.* This follows from proposition I.2 and theorem I.25. □

Let $F' \to F$ be a morphism of strict presheaves of groupoids. Let $F'_\bullet$ be the simplicial presheaf of groupoids such that $F'_i := F' \times_F F' \times_F \cdots \times_F F'$ ($i+1$ times). Let $Ner(F'_\bullet)$ be the bi-simplicial presheaf such that $Ner(F'_\bullet)_{\bullet,i} := Ner(F'_i)$. Let $diag(Ner(F'_\bullet))$ be the diagonal.

**Lemma II.3.** *Let $p : F' \to F$ be an étale, representable, surjective morphsim of Deligne-Mumford stacks (here stacks are strict presheaves of groupoids). Then the canonical morphism*

$$diag(Ner(F'_\bullet)) \to Ner(F)$$

*is an étale local weak equivalence.*

*Proof.* Let $U \to F$ be an atlas and $U_\bullet$ be the associated Čech simplicial scheme. Let $U'_{\bullet,\bullet}$ be the bi-simplicial algebraic space such that $U'_{\bullet,i} := U_\bullet \times_F F'_i$ for $i \geq 0$. Hence, $U'_{j,\bullet} := U_j \times_F F'_\bullet$ for $j \geq 0$. There are natural morphisms $diag(U'_{\bullet,\bullet}) \to diag(Ner(F'_\bullet))$ and $diag(U'_{\bullet,\bullet}) \to U_\bullet$. For $i, j \geq 0$, $U'_{\bullet,i} \to F'_i$ and $U'_{j,\bullet} \to U_j$ are étale Čech hypercovering, hence $U'_{\bullet,i} \to Ner(F'_i)$ and $U'_{j,\bullet} \to U_j$ are étale local weak equivalences. By [14, XII.3.3]

$$diag(U'_{\bullet,\bullet}) \cong hocolim_{n\in\triangle}(U'_{\bullet,n}) \cong diag(Ner(F'_\bullet))$$

and

$$diag(U'_{\bullet,\bullet}) \cong hocolim_{n\in\triangle}(U'_{n,\bullet}) \cong U_\bullet.$$

This proves the lemma. $\square$

## 2.2 Motives of Deligne-Mumford Stacks, I

In this section, we first show that the motive of a separated Deligne-Mumford stack is naturally isomorphic to the motive of its coarse moduli space. We also prove blow-up and projective bundle formulas for smooth Deligne-Mumford stacks. We end the section with the construction of the Gysin triangle associated with a smooth closed substack $Z$ of a smooth Deligne-Mumford stack $F$.

### 2.2.1 Motive of coarse moduli space

Let $X$ be a scheme and let $G$ be a group acting on $X$. Then $G$ acts on the presheaf $\mathbb{Q}_{tr}(X)$. Let $\mathbb{Q}_{tr}(X)_G$ be the $G$-coinvariant presheaf, such that for any $Y \in Sm/k$ we have $\mathbb{Q}_{tr}(X)_G(Y) := (\mathbb{Q}_{tr}(X)(Y))_G$.

**Lemma II.4.** *Let $X$ be a smooth quasi-projective scheme and let $G$ be a finite group acting on $X$. Let $[X/G]$ be the quotient Deligne-Mumford stack and $X/G$ be the quotient scheme. Then*

1. $M([X/G]) \cong \mathbb{Q}_{tr}(X)_G$ *in* $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$*;*
2. $\mathbb{Q}_{tr}(X/G) \cong \mathbb{Q}_{tr}(X)_G$ *as presheaves.*

*Hence the canonical morphism $M([X/G]) \to \mathbb{Q}_{tr}(X/G)$ is an isomorphism in* $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$.

*Proof.* To deduce (1), we observe that the morphism $X \to [X/G]$ sending $X$ to the trivial $G$-torsor $X \times G \to X$ is an étale atlas. Let $X_\bullet$ be the corresponding Čech simplicial scheme. Then $\mathbb{Q}_{tr}(X_\bullet) \cong M([X/G])$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$. Moreover, $\mathbb{Q}_{tr}(X_\bullet)(Y) \cong (\mathbb{Q}_{tr}(X)(Y) \otimes \mathbb{Q}[EG])/G$. Hence the complex $\mathbb{Q}_{tr}(X_\bullet)(Y)$ computes the homology of $G$ with coefficient in the $G$-module $\mathbb{Q}_{tr}(X_\bullet)(Y)$. Since $G$ is finite and we work with rational coefficients, we have $\mathbb{Q}_{tr}(X_\bullet)(Y) \cong (\mathbb{Q}_{tr}(X)(Y))_G$ in the derived category of chain complexes of $\mathbb{Q}$-vector spaces.

To deduce (2), we observe that the canonical quotient morphism $\pi : X \to X/G$ is finite and surjective. Let $m$ be the generic degree of $\pi$, then the morphism $\Gamma_\pi : \mathbb{Q}_{tr}(X) \to \mathbb{Q}_{tr}(X/G)$ has a section $\frac{1}{m}{}^t\Gamma_\pi$. Hence $\mathbb{Q}_{tr}(X/G)$ is isomorphic to the image of the projector $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi$. But $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi = \frac{1}{|G|}\sum_{g\in G} g$ whose image is isomorphic to $\mathbb{Q}_{tr}(X)_G$. □

**Remark II.5.** *In the proof above, the composition $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi$ is well defined (cf. [47, Definition 1A.11]). Indeed, since $X/G$ is normal, the finite correspondence ${}^t\Gamma_\pi$ is a relative cycle over $X/G$ by [47, Theorem 1A.6].*

**Theorem II.6.** *Let $F$ be a separated smooth Deligne-Mumford stack over a field $k$ of characteristic $0$. Let $\pi : F \to X$ be the coarse moduli space. Then the natural morphism $M(\pi) : M(F) \to \mathbb{Q}_{tr}(X)$ is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.*

*Proof.* By theorem II.16, there exists an étale covering $(U_i)_{i\in I}$ of $X$, such that $U_i \cong X_i/H_i$ and $F_i := U_i \times_X F \cong [X_i/H_i]$ for quasi-projective smooth schemes $X_i$ and finite groups $H_i$. Let $F' := \coprod F_i$ and $X' := \coprod X_i/H_i$. Then by lemma II.3, $M(diag(Ner(F'_\bullet))) \cong M(F)$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Similarly, $\mathbb{Q}_{tr}(X'_\bullet) \cong \mathbb{Q}_{tr}(X)$. To show that $M(F) \cong \mathbb{Q}_{tr}(X)$, it is then enough to show that $M(Ner(F'_n)) \cong M(X'_n)$. Hence we are reduced to the case $F = [X/G]$ which follows from lemma II.4. □

#### 2.2.2 Motive of a projective bundle

Let $\mathcal{E}$ be a vector bundle of rank $n+1$ on a smooth finite type Deligne-Mumford stack $F$ and let $Proj(\mathcal{E})$ denote the associated projective bundle over $F$.

**Theorem II.7.** *There exists a canonical isomorphism in* $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$*:*

$$M(Proj(\mathcal{E})) \to \bigoplus_{i=0}^{n} M(F) \otimes \mathbb{Q}(i)[2i].$$

*Proof.* Let $a : U \to F$ be an atlas of $F$ and $V := Proj(a^*(\mathcal{E})) \to Proj(\mathcal{E})$ be the induced atlas of $Proj(\mathcal{E})$.

The line bundle $\mathcal{O}_{Proj(\mathcal{E})}(1)$ induces a canonical map

$$\tau : M(Proj(\mathcal{E})) \to \mathbb{Q}(1)[2]$$

in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$ by corollary II.24 below. Here we take $\mathbb{Q}(1)[2] := C_*(\mathbb{P}^1,\infty) = N(\underline{Hom}(\triangle^\bullet,\mathbb{Q}_{tr}(\mathbb{P}^1,\infty)))$ where $N$ is the normalized chain complex and $\underline{Hom}$ is the internal $Hom$ (see [47, page 15-16] for $\mathbb{Q}_{tr}(\mathbb{P}^1,\infty)$ and $C_*$). As the complex $C_*(\mathbb{P}^1,\infty)$ is fibrant for the projective motivic model structure (see [6, Corollary 2.155]), $\tau$ is represented by a morphism

$$\tau' : N(\mathbb{Q}_{tr}(V_\bullet)) \to C_*(\mathbb{P}^1,\infty)$$

in $K(PST(k))$ where $V_\bullet$ is the Čech complex associated to the atlas $a : V \to Proj(\mathcal{E})$. By the Dold-Kan correspondence we get a morphism

$$\tau' : \mathbb{Q}_{tr}(V_\bullet) \to \underline{Hom}(\triangle^\bullet,\mathbb{Q}_{tr}(\mathbb{P}^1,\infty)) \tag{2.2.1}$$

in $\triangle^{op}(PST(k))$. Note that in simplicial degree zero, the induced map $\mathbb{Q}_{tr}(V) \to \mathbb{Q}_{tr}(\mathbb{P}^1,\infty)$ represents the class of $\mathcal{O}_{Proj(\mathcal{E}|_V)}(1)$. Using the commutativity of

$$\begin{tikzcd}
\mathbb{Q}_{tr}(V_i) \arrow[r] \arrow[d] & \underline{Hom}(\triangle^i,\mathbb{Q}_{tr}(\mathbb{P}^1,\infty)) \arrow[d, "we"] \\
\mathbb{Q}_{tr}(V) \arrow[r] & \mathbb{Q}_{tr}(\mathbb{P}^1,\infty)
\end{tikzcd}$$

the upper horizontal morphism also represents the class of $\mathcal{O}_{Proj(\mathcal{E}|_{V_i})}(1)$ modulo the $\mathbb{A}^1$-weak equivalence $we$.

The morphism of simplicial presheaves (2.2.1) induces a morphism

$$(\tau')^m : \mathbb{Q}_{tr}(\underbrace{V_\bullet \times \cdots \times V_\bullet}_{m\,\text{times}}) \to \underline{Hom}(\underbrace{\triangle^\bullet \times \cdots \times \triangle^\bullet}_{m\,\text{times}}, \mathbb{Q}_{tr}(\mathbb{P}^1, \infty)^{\wedge m})$$

between multisimplicial presheaves with transfers for every positive integer $m$. The diagonals $\triangle^\bullet \to diag(\triangle^\bullet \times \cdots \times \triangle^\bullet)$ and $V_\bullet \to diag(V_\bullet \times \cdots \times V_\bullet)$ give a morphism

$$(\tau')^m : \mathbb{Q}_{tr}(V_\bullet) \to \underline{Hom}(\triangle^\bullet, \mathbb{Q}_{tr}(\mathbb{P}^1, \infty)^{\wedge m}).$$

Moreover the morphism $\mathbb{Q}_{tr}(V_\bullet) \to \mathbb{Q}_{tr}(U_\bullet)$ gives a morphism

$$\sigma : \mathbb{Q}_{tr}(V_\bullet) \to diag(\bigoplus_{m=0}^{n} \underline{Hom}(\triangle^\bullet, \mathbb{Q}_{tr}(\mathbb{P}^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_\bullet)))$$

of simplicial persheaves with transfers. Here $U_\bullet$ is the associated Čech complex of $a : U \to F$.

In degree $i$ the morphism $\sigma$ coincides with the one from [47, Construction 15.10] modulo the $\mathbb{A}^1$-weak equivalence

$$\bigoplus_{m=0}^{n} \underline{Hom}(\triangle^i, \mathbb{Q}_{tr}(\mathbb{P}^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_i)) \to \bigoplus_{m=0}^{n} \mathbb{Q}_{tr}(\mathbb{P}^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_i).$$

It follows from [47, Theorem 15.12] that $\sigma$ induces $\mathbb{A}^1$-weak equivalence after passing to the normalized complex. This proves the theorem. □

#### 2.2.3 Motives of blow-ups

Let $X$ be a $k$-scheme. Let $X' \to X$ be a blow-up with center $Z$ and $Z' := Z \times_X X'$ be the exceptional divisor. Then [47, Theorem 13.26] can be rephrased as follows. (Recall that $char(k) = 0$.)

**Theorem II.8.** *The following commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}_{tr}(Z') & \longrightarrow & \mathbb{Z}_{tr}(X') \\ \downarrow & & \downarrow \\ \mathbb{Z}_{tr}(Z) & \longrightarrow & \mathbb{Z}_{tr}(X) \end{array}$$

*is homotopy co-cartesian (with respect to the étale $\mathbb{A}^1$-local model structure).*

**Proposition II.9.** *Let $F$ be a smooth Deligne-Mumford stack and $Z \subset F$ be a smooth closed substack. Let $Bl_Z(F)$ be the blow-up of $F$ with center $Z$ and $E := Z \times_F Bl_Z(F)$ be the exceptional divisor. Then one has a canonical distinguished triangle of the form :*

$$M(E) \to M(Z) \oplus M(Bl_Z(F)) \to M(F) \to M(E)[1].$$

*Proof.* Let $a : U \to F$ be an atlas and let $U_\bullet$ be the associated Čech complex. Then the following square of simplicial presheaves with transfers

$$\begin{CD} \mathbb{Q}_{tr}(U_\bullet \times_F E) @>>> \mathbb{Q}_{tr}(U_\bullet \times_F Bl_Z(F)) \\ @VVV @VVV \\ \mathbb{Q}_{tr}(U_\bullet \times_F Z) @>>> \mathbb{Q}_{tr}(U_\bullet) \end{CD}$$

is homotopy co-cartesian in each degree by theorem II.8. Since homotopy colimits commutes with homotopy push-outs, the following square

$$\begin{CD} M(E) @>>> M(Bl_Z(F)) \\ @VVV @VVV \\ M(Z) @>>> M(F) \end{CD}$$

is homotopy co-cartesian and hence we get our result. □

**Theorem II.10.** *Let $F$ be a smooth Deligne-Mumford stack and $Z \subset F$ be a smooth closed substack of pure codimension $c$. Let $Bl_Z(F)$ be the blow-up of $F$ with center $Z$. Then*

$$M(Bl_Z(F)) \cong M(F) \bigoplus (\oplus_{i=1}^{c-1} M(Z)(i)[2i]).$$

*Proof.* By proposition II.9 we have a canonical distinguished triangle

$$M(p^{-1}(Z)) \to M(Z) \oplus M(Bl_Z(F)) \to M(F) \to M(p^{-1}(Z))[1],$$

where $p : Bl_Z(F) \to F$ is the blow-up. Since $Z$ is smooth, $p^{-1}(Z) \cong Proj(N_Z(F))$, where $N_Z(F)$ is the normal bundle. Hence using theorem II.7, it is enough to show that the morphism $M(F) \to M(p^{-1}(Z))[1]$ is zero in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$. Let $q : Bl_{Z\times\{0\}}(F\times\mathbb{A}^1) \to F\times\mathbb{A}^1$ be the blow-up of $Z\times\{0\}$ in $F\times\mathbb{A}^1$.

Following the proof of [62, Proposition 3.5.3], consider the morphism of exact triangles:

$$\begin{tikzcd}
M(p^{-1}(Z)) \arrow[r] \arrow[d] & M(q^{-1}(Z\times\{0\}) \arrow[d] \\
M(Z)\oplus M(Bl_Z F) \arrow[r] \arrow[d] & M(Z\times\{0\})\oplus M(Bl_{Z\times\{0\}}F\times\mathbb{A}^1) \arrow[d, "f"] \\
M(F) \arrow[r, "s_0"] \arrow[d, "g"] & M(F\times\mathbb{A}^1) \arrow[d, "h"] \\
M(p^{-1}(Z))[1] \arrow[r, "a"] & M(q^{-1}(Z\times\{0\}))[1]
\end{tikzcd}$$

Since the morphism $s_0$ is an isomorphism and since by theorem II.7 $a$ is split injective, the morphism $g$ is zero if $h$ is zero. To show that $h$ is zero it is enough to show that $f$ has a section. This is the case as the composition

$$M(F\times\{1\}) \to M(Bl_{Z\times\{0\}}F\times\mathbb{A}^1) \to M(F\times\mathbb{A}^1)$$

is an isomorphism. $\square$

#### 2.2.4 Gysin triangle

Given a morphism $F \to F'$ of Deligne-Mumford stacks, let

$$M\left(\frac{F'}{F}\right) := cone(M(F)\to M(F')).$$

Similarly given a morphism $V_\bullet \to U_\bullet$ of simplicial schemes, let

$$\mathbb{Q}_{tr}\left(\frac{U_\bullet}{V_\bullet}\right) := cone(\mathbb{Q}_{tr}(V_\bullet)\to\mathbb{Q}_{tr}(U_\bullet)).$$

**Lemma II.11.** *Let $f : F' \to F$ be an étale morphism of smooth Deligne-Mumford stacks, and let $Z \subset F$ be a closed substack such that $f$ induces an isomorphism*

*$f^{-1}(Z) \cong Z$. Then the canonical morphism*

$$M\left(\frac{F'}{F'-Z}\right) \to M\left(\frac{F}{F-Z}\right)$$

*is an isomorphism.*

*Proof.* Let $v' : V' \to F'$ be an atlas of $F'$, and let $v : V \to F - Z$ be an atlas of the complement of $Z$. Then $U = V \coprod V' \to F$ is an atlas of $F$. Let $f_\bullet : V'_\bullet \to U_\bullet$ be the induced morphism between the associated Čech simplicial schemes. In each simplicial degree $i$, we have an étale morphism $f_i : V'_i \to U_i$ such that $f_i$ induces an isomorphism $Z \times_F V'_i \cong Z \times_F U_i$. Let $Z_\bullet := Z \times_F U_\bullet \cong Z \times_F V'_\bullet$. It is enough to show that the canonical morphism $M\left(\frac{V_\bullet}{V_\bullet - Z_\bullet}\right) \to M\left(\frac{U_\bullet}{U_\bullet - Z_\bullet}\right)$ is an isomorphism. This is indeed the case as $\mathbb{Q}_{tr}\left(\frac{V_\bullet}{V_\bullet - Z_\bullet}\right) \cong \mathbb{Q}_{tr}\left(\frac{U_\bullet}{U_\bullet - Z_\bullet}\right)$ by [61, Proposition 5.18]. □

**Lemma II.12.** *Let $p : V \to F$ be a vector bundle of rank $d$ over a smooth Deligne-Mumford stack $F$. Let $s : F \to V$ be the zero section of $p$. Then*

$$M\left(\frac{V}{V \setminus s}\right) \cong M(F)(d)[2d].$$

*Proof.* Using lemma II.11, we have an isomorphism

$$M\left(\frac{V}{V \setminus s}\right) \cong M\left(\frac{Proj(V \oplus O)}{Proj(V \oplus O) \setminus s}\right).$$

The image of the embedding $Proj(V) \to Proj(V \oplus O)$ is disjoint from $s$ and $\iota : Proj(V) \to Proj(V \oplus O) \setminus s$ is the zero section of a line bundle. Thus, the induced morphism $\iota : M(Proj(V)) \to M(Proj(V \oplus O) \setminus s)$ is an $\mathbb{A}^1$-weak equivalence. (This can be checked using an explicit $\mathbb{A}^1$-homotopy as in the classical case where the base is a scheme.) It follows that

$$M\left(\frac{V}{V \setminus s}\right) \cong M\left(\frac{Proj(V \oplus O)}{Proj(V)}\right).$$

Now using theorem II.7, we get

$$M\left(\frac{Proj(V\oplus O)}{Proj(V)}\right)\cong M(F)(d)[2d].$$

This proves the lemma. □

**Theorem II.13.** *Let $Z\subset F$ be a smooth closed codimension $c$ substack of a smooth Deligne-Mumford stack $F$. Then there exists a Gysin exact triangle:*

$$M(F\setminus Z)\to M(F)\to M(Z)(c)[2c]\to M(F\setminus Z)[1].$$

*Proof.* We have the following obvious exact triangle

$$M(F\setminus Z)\xrightarrow{i} M(F)\to M\left(\frac{F}{F\setminus Z}\right)\to M(F\setminus Z)[1].$$

We need to show that $M\left(\frac{F}{F\setminus Z}\right)\cong M(Z)(c)[2c]$ in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$. Let $D_Z(F)$ be the space of deformation to the normal cone and let $N_Z(F)$ be the normal bundle. Consider the following commutative diagram of stacks:

$$\begin{tikzcd}
Z\times 1 \arrow[r] \arrow[d, hook] & Z\times\mathbb{A}^1 \arrow[d, hook] & Z\times 0 \arrow[l] \arrow[d, hook, "s_Z"] \\
F\times 1 \arrow[r, hook] \arrow[d] & D_Z(F) \arrow[d] & N_Z(F) \arrow[l, hook'] \arrow[d] \\
1 \arrow[r] & \mathbb{A}^1 & 0. \arrow[l]
\end{tikzcd}$$

This gives morphisms

$$s^1: M\left(\frac{F}{F\setminus Z}\right)\to M\left(\frac{D_Z(F)}{D_Z(F)\setminus(Z\times\mathbb{A}^1)}\right)$$

and

$$s^0: M\left(\frac{N_Z(F)}{N_Z(F)\setminus s_Z}\right)\to M\left(\frac{D_Z(F)}{D_Z(F)\setminus(Z\times\mathbb{A}^1)}\right).$$

Let $U\to F$ be an atlas of $F$ and let $U_\bullet$ be the associated Čech simplicial scheme. Then $s^1$ can be described as

$$s^1:\mathbb{Q}_{tr}\left(\frac{U_\bullet}{(F\setminus Z)\times_F U_\bullet}\right)\to\mathbb{Q}_{tr}\left(\frac{D_Z(F)\times_F U_\bullet}{(D_Z(F)\setminus(Z\times\mathbb{A}^1))\times_F U_\bullet}\right).$$

Let $Z_i := Z \times_F U_i$. In each simplicial degree $i$ the morphism $(s^1)_i : \mathbb{Q}_{tr}\left(\frac{U_i}{U_i\backslash(Z_i)}\right) \to \mathbb{Q}_{tr}\left(\frac{D_{Z_i}(U_i)}{D_{Z_i}(U_i)\backslash(Z_i\times\mathbb{A}^1)}\right)$ induced by $s^1$ is an $\mathbb{A}^1$-weak equivalence by lemma II.14. Hence $s^1$ is an $\mathbb{A}^1$-weak equivalence. Similarly, $s^0$ is an $\mathbb{A}^1$-weak equivalence. Hence we get an isomorphism $M\left(\frac{F}{F\backslash Z}\right) \cong M\left(\frac{N_Z(F)}{N_Z(F)\backslash s_Z}\right)$. But $M\left(\frac{N_Z(F)}{N_Z(F)\backslash s_Z}\right) \cong M(Z)(c)[2c]$ by lemma II.12. □

**Lemma II.14.** *Suppose we have a cartesian diagram of smooth schemes*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "0"] & Y \arrow[d, "u"] \\ \mathbb{A}^1 \times Z \arrow[r, "v"] & X \end{tikzcd}$$

*where $u$ and $v$ are closed embeddings and $Y$ has codimension 1 in $X$. Then the canonical morphism $M\left(\frac{Y}{Y\backslash Z}\right) \to M\left(\frac{X}{X\backslash(\mathbb{A}^1\times Z)}\right)$ is an isomorphism.*

*Proof.* Let $c$ be the codimension of $Z$ in $Y$; it is also the codimension of $\mathbb{A}^1 \times Z$ in $X$. Using [62, Proposition 3.5.4] we have $M\left(\frac{Y}{Y\backslash Z}\right) \cong M(Z)(c)[2c]$ and $M\left(\frac{X}{X\backslash(\mathbb{A}^1\times Z)}\right) \cong M(Z \times \mathbb{A}^1)(c)[2c]$. Since $M(Z) \cong M(Z \times \mathbb{A}^1)$, we get the lemma. □

## 2.3 Motives of Deligne-Mumford stacks, II

The main goal of this section is to show that the motive of a smooth Deligne-Mumford stack $F$ is a direct factor of the motive of a smooth and quasi-projective variety. Moreover, if $F$ is proper, this variety can be chosen to be projective.

### 2.3.1 Blowing-up Deligne-Mumford stacks and principalization

Let $F$ be a smooth Deligne-Mumford stack and let $Z$ be a closed substack of $F$. The blow-up of $F$ along $Z$ is a Deligne-Mumford stack $Bl_Z(F)$ together with a representable projective morphism $\pi : Bl_Z(F) \to F$.

**Theorem II.15.** *Let $F$ be a smooth Deligne-Mumford stack of finite type over a field of characteristic zero. Let $\mathcal{O}_F$ be the structure sheaf and let $\mathcal{I} \subset \mathcal{O}_F$ be a coherent ideal. Then there is a sequence of blow-ups in smooth centers*

$$\pi : F_r \xrightarrow{\pi_r} F_{r-1} \xrightarrow{\pi_{r-1}} \dots \xrightarrow{\pi_1} F$$

*such that $\pi^*\mathcal{I} \subset \mathcal{O}_{F_r}$ is locally principal.*

*Proof.* Let $a : U \to F$ be an atlas and denote $\mathcal{J} := a^*\mathcal{I}$. By Hironaka's resolution of singularities [43, Theorem 3.15], we have a sequence of blow-ups in smooth centers

$$\pi' : U_r \xrightarrow{\pi'_r} U_{r-1} \xrightarrow{\pi'_{r-1}} \dots \xrightarrow{\pi'_1} U,$$

such that $(\pi')^*\mathcal{J}$ is a locally principal coherent ideal on $U_r$. Moreover this sequence commutes with arbitary smooth base change. Hence the sequence $\pi'$ descends to give the sequence of the statement. $\square$

**Lemma II.16.** *Let $F' \to F$ be a (quasi-)projective representable morphism of Deligne-Mumford stacks. Let $X$ and $X'$ be the coarse moduli spaces of $F$ and $F'$ respectively. Then the induced morphism $X' \to X$ is (quasi-)projective. In particular, if $X$ is (quasi-)projective then so is $X'$.*

*Proof.* [45, lemma 2, Theorem 1]. $\square$

The proof of the following theorem was communicated to us by David Rydh.

**Theorem II.17.** *Given a smooth finite type Deligne-Mumford stack $F$ over $k$, there exists a sequence of blow-ups in smooth centers $\pi : F' \to F$, such that the coarse moduli space of $F'$ is quasi-projective.*

*Proof.* Let $p : F \to X$ be the morphism to the coarse moduli space of $F$. $X$ is a separated algebraic space. By Chow's Lemma ([41, Theorem 3.1]) we have a projective morphism $g : X' \to X$ from a quasi-projective scheme $X'$. Moreover by [26, Corollary 5.7.14] we may assume that $g$ is a blow-up along a closed subspace $Z \subset X$. Let $F' := X' \times_X F$. There is a morphism $p' : F' \to X'$. Since $F$ is tame $X'$ is the coarse moduli space of $F'$ ([1, Cor 3.3]). Let $T := Z \times_X F$ and let $\pi : Bl_T(F) \to F$ be the blow-up of $F$ along $T$. Then $Bl_T(F)$ is the closure of $F \setminus T$ in $F'$. As $F'$ is tame the coarse moduli space of $Bl_T(F)$ is a closed subscheme of $X'$. Hence $Bl_T(F)$ has quasi-projective coarse moduli space.

Now by II.15 We have a sequence of blow-ups in smooth centers $\pi : F_r \to F$ such that the ideal sheaf defining $T$ is principalized. Hence there exists a canonical projective representable morphism $\pi' : F_r \to Bl_T(F)$. Since $Bl_T(F)$ has quasi-projective coarse moduli space and $\pi'$ is a projective representable morphism, we have our result by lemma II.16. □

#### 2.3.2 Chow motives and motives of proper Deligne-Mumford stack

Let $f : X \to Y$ be a finite morphism between smooth schemes such that each connected component of $X$ maps surjectively to a connected component of $Y$ and generically over $Y$ the degree of $f$ is constant equal to $m$. Then the transpose of $\Gamma_f$ is a correspondence from $Y$ to $X$. This defines a morphism ${}^t f : \mathbb{Q}_{tr}(Y) \to \mathbb{Q}_{tr}(X)$ such that $f \circ (\frac{1}{m}{}^t f)$ is the identity.

**Remark II.18.** *Suppose we are given a cartesian diagram of smooth schemes*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "h'"] & X' \arrow[d, "h"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

*with $h$ étale. Assume that $f$ is a finite morphism such that each connected component of $Y$ maps surjectively to a connected component of $X$ and generically over $X$ the degree of $f$ is constant equal to $m$. Then $f'$ satisfies the same properties as $f$.*

*Thus we have morphisms* ${}^t f : \mathbb{Q}_{tr}(X) \to \mathbb{Q}_{tr}(Y)$ *and* ${}^t f' : \mathbb{Q}_{tr}(X') \to \mathbb{Q}_{tr}(Y')$. *Using the definition of composition of finite correspondences one can easily verify that* $(h') \circ ({}^t f') = ({}^t f) \circ (h)$.

**Lemma II.19.** *Let $F$ be a smooth Deligne-Mumford stack. Assume that there exists a smooth scheme $X$ and a finite surjective morphism $g : X \to F$. Then $M(F)$ is a direct factor of $M(X)$.*

*Proof.* We may assume that $F$ and $X$ are connected. Let $a : U \to F$ be an atlas and $U_\bullet$ the associated Čech complex. Set $V_\bullet := U_\bullet \times_F X$. Then $g'_\bullet : V_\bullet \to U_\bullet$ is finite and surjective of constant degree $m$ in each simplicial degree. It follows from II.18 that ${}^t g'_\bullet : \mathbb{Q}_{tr}(U_\bullet) \to \mathbb{Q}_{tr}(V_\bullet)$ is a morphism of simplicial sheaves with transfers such that $g'_\bullet \circ (\frac{1}{m} {}^t g'_\bullet) = id$. Hence $\mathbb{Q}_{tr}(U_\bullet)$ is a direct factor of $\mathbb{Q}_{tr}(V_\bullet)$.

Since $V_\bullet$ is a Čech resolution of $X$, we have $\mathbb{Q}_{tr}(V_\bullet) \cong \mathbb{Q}_{tr}(X)$ by [47, Proposition 6.12]. This proves the result. □

**Theorem II.20.** *Let $F$ be a proper (resp. not necessarily proper) smooth Deligne-Mumford stack. Then $M(F)$ is a direct summand of the motive of a projective (resp. quasi-projective) variety.*

*Proof.* We can assume that $F$ is connected. By II.17 we get a sequence of blow-ups with smooth centers $\pi : F' \to F$ such that $F'$ has (quasi)-projective coarse moduli space. By II.10 $M(F)$ is a direct summand of $M(F')$. By [45, Theorem 1] there exists a smooth (quasi)-projective variety $X$ and a finite flat morphism $g : X \to F'$. Hence $M(F')$ is a direct summand of $M(X)$ which proves our claim. □

Recall that the category of effective geometric motives $\mathbf{DM}^{eff}_{gm}(k, \mathbb{Q})$ is the thick subcategory of $\mathbf{DM}^{eff}(k, \mathbb{Q})$ generated by the motives $M(X)$ for $X \in Sm/k$ (see [47, Definition 14.1]).

**Corollary II.21.** *For any smooth finite type Deligne-Mumford stack $F$, $M(F)$ is an effective geometric motive.*

**Remark II.22.** *By [47, Proposition 20.1] the category of effective Chow motives embeds into $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$. Theorem II.20 shows that $M(F)$ lies in the essential image of this embedding for any smooth proper Deligne-Mumford stack $F$.*

## 2.4 Motivic cohomology of stacks

**Lemma II.23.** *Let $F$ be a Deligne-Mumford stack. We have an isomorphism*

$$H^{2i-n}_M(F, i) \simeq Hom_{\mathbf{DM}^{\text{eff}}(k,\mathbb{Q})}(M(F), \mathbb{Q}(n)[2i-n]).$$

*Proof.* Let $U \to F$ be an atlas and $U_\bullet$ be the associated Čech complex. We have an étale weak equivalence $\mathbb{Q}(U_\bullet) \to \mathbb{Q}$ of complexes of sheaves on $F_{\acute{e}t}$. Here $\mathbb{Q}$ is the constant sheaf on $F_{\acute{e}t}$. Writing $D(F_{\acute{e}t})$ for the derived category of sheaves of $\mathbb{Q}$-vector spaces on $F_{\acute{e}t}$, we thus have

$$H^{2i-n}_M(F, i) \cong Hom_{D(F_{\underline{\acute{e}t}})}(\mathbb{Q}(U_\bullet), \mathbb{Q}(i)[2i-n]).$$

Let $a : \mathbb{Q}(i) \to L$ be a fibrant replacement for the injective local model structure on $K(PST(k))$ and let $b : L|_{F_{\acute{e}t}} \to M$ be a fibrant replacement for the injective local model structure on $K(F_{\acute{e}t})$.

Since both $a$ and $b$ are étale local weak equivalences the composition $b \circ a : \mathbb{Q}(i)|_{F_{\acute{e}t}} \to M$ is an étale weak equivalence. It follows that

$$Hom_{D(F_{\acute{e}t})}(\mathbb{Q}(U_\bullet), (\mathbb{Q}(i)|_{F_{\acute{e}t}})[2i-n]) \cong Hom_{Ho(K(F_{\acute{e}t}))}(\mathbb{Q}(U_\bullet), M[2i-n]).$$

Using [63, 2.7.5], it follows that $H^{2i-n}_M(F, i)$ is the $(2i-n)$-th cohomology of the complex $Tot(Hom(\mathbb{Q}(U_\bullet), M))$.

On the other hand, since $a$ is an étale weak equivalence and $\mathbb{Q}(i)$ is $\mathbb{A}^1$-local, $L$ is also $\mathbb{A}^1$-local. It follows that

$$Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})}(\mathbb{Q}_{tr}(U_\bullet), \mathbb{Q}(i)[2i-n]) \cong Hom_{Ho(K(PST(k)))}(\mathbb{Q}_{tr}(U_\bullet), L[2i-n]).$$

Again by ([63, 2.7.5]), the right hand side is same as $(2i-n)$-th cohomology of the complex $Tot(Hom(\mathbb{Q}_{tr}(U_\bullet), L))$.

To prove the lemma it is now sufficient to show that $L(X) \to M(X)$ is a quasi-isomorphism for any smooth $k$-scheme $X$. By definition

$$H^n(L(X)) \cong H^n(Hom(\mathbb{Q}_{tr}(X), F)) \cong \mathbb{E}xt^n(\mathbb{Q}_{tr}(X), \mathbb{Q}(i)[n])$$

and by [47, 6.25] we have

$$\mathbb{E}xt^n(\mathbb{Q}_{tr}(X), \mathbb{Q}(i)[n]) = H^n_{\acute{e}t}(X, \mathbb{Q}(i))$$

which is same as $H^n(M(X))$. □

**Corollary II.24.** *Let $F$ be a Deligne-Mumford stack and let $\mathcal{O}_F$ be the structure sheaf. Then we have an isomorphism*

$$Pic(F) \otimes \mathbb{Q} \cong H^1_{\acute{e}t}(F, \mathcal{O}_F^\times \otimes \mathbb{Q}) \cong Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})}(M(F), \mathbb{Q}(1)[2])$$

*Proof.* The first isomorphism follows from [52, page 65, 67]. By [47, Theorem 4.1] $\mathcal{O}_F^*[1] \otimes \mathbb{Q} \cong \mathbb{Q}(1)[2]$. So the second isomorphism is a particular case of lemma II.23.

□

**Remark II.25.** *From the proofs, it is easy to see that Lemma II.23 and Corollary II.24 are true integrally if we use Voevodsky's category of étale motives with integral coefficients $\mathbf{DM}^{\mathit{eff},\, \acute{e}t}(k, \mathbb{Z})$.*

## 2.5 Chow motives of stacks and comparisons

Let $C$ be a symmetric monoidal category and let $X \in Ob(C)$. Recall that an object $Y \in C$ is called a strong dual of $X$ if there exist two morphisms $coev : \mathbb{1} \to Y \otimes X$ and $ev : X \to \mathbb{1}$, such that the composition of

$$X \xrightarrow{id \otimes coev} X \otimes Y \otimes X \xrightarrow{ev \otimes id} X \tag{2.5.1}$$

and the composition of

$$Y \xrightarrow{coev \otimes id} Y \otimes X \otimes Y \xrightarrow{id \otimes ev} Y \tag{2.5.2}$$

are identities.

**Lemma II.26.** *Let $F$ be a proper smooth Deligne-Mumford stack of pure dimension $d$. Then $h_{DM}(F) := (F, \Delta_F, 0)$ has a strong dual in $\mathcal{M}_k^{DM}$. It is given by $(F, \Delta_F, -d)$.*

*Proof.* Set $h_{DM}(F)^* := (F, \Delta_F, -d)$. We need to give morphisms $coev : \mathbb{1} \to h_{DM}(F)^* \otimes h_{DM}(F)$ and $ev : h_{DM}(F) \otimes h_{DM}(F)^* \to \mathbb{1}$, such that (2.5.1) and (2.5.2) are satisfied. The morphisms $coev$ and $ev$ are given by $\Delta_F \in A^d(F \times F)$. To compute the composition of (2.5.1), we observe that intersection of the cycles $\Delta_F \times \Delta_F \times F$ and $F \times \Delta_F \times \Delta_F$ in $F \times F \times F \times F \times F$ is equal to $\delta(F)$ where $\delta : F \to F \times F \times F \times F \times F$ is the diagonal morphism. The push-forward to $F \times F$ of the latter is simply the diagonal of $F \times F$. This shows that the composition of (2.5.1) is the identity of $h_{DM}(F)$. The composition of (2.5.2) is treated using the same method. $\square$

By [58, Theorem 2.1] the natural functor $e : \mathcal{M}_k \to \mathcal{M}_k^{DM}$ is an equivalence of $\mathbb{Q}$-linear tensor categories. This equivalence preserves the subcategories of effective motives. Thus, after inverting this equivalence we can associate an effective Chow motive $h(F) \in \mathcal{M}_k^{eff}$ to every smooth and proper Deligne-Mumford stack $F$. On the other hand, by [47, Proposition 20.1] there exists a fully faithful functor $\iota : \mathcal{M}_k^{\mathit{eff}} \to \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$.

**Theorem II.27.** *Let $F$ be a smooth proper Deligne-Mumford stack. Then $M(F) \cong \iota \circ h(F)$.*

*Proof.* We may assume that $F$ has pure dimension $d$. By II.20 $M(F)$ is a direct factor of the motive of a smooth and projective variety $W$ such that $dim(W) = d$. By [47, Example 20.11],

$$\underline{Hom}(M(W), \mathbb{Q}(d)[2d]) \cong M(W)$$

is an effective Chow motive. It follows that $\underline{Hom}(M(F), \mathbb{Q}(d)[2d])$ is also an effective Chow motive.

We first show that $\iota \circ h(F) \cong \underline{Hom}(M(F), \mathbb{Q}(d)[2d])$. Let $\mathcal{V}_k$ be the category of smooth and projective varieties over $k$. For $M \in \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$ denote $\omega_M$ the presheaf on $\mathcal{V}_k$ defined by

$$X \in \mathcal{V}_k \mapsto Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})}(M(X), M).$$

Using V.1, it is enough to construct an isomorphism of presheaves

$$\omega_{\underline{Hom}(M(F),\mathbb{Q}(d)[2d])} \cong \omega_{\iota \circ h(F)}.$$

The right hand side is by definition the presheaf $A^{dim(F)}(- \times F)$. For $X \in \mathcal{V}_k$, we have

$$\begin{aligned} \omega_{\underline{Hom}(M(F),\mathbb{Q}(d)[2d])}(X) &= \hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})}(M(X), \underline{Hom}(M(F), \mathbb{Q}(d)[2d])) \\ &= \hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})}(M(X \times F), \mathbb{Q}(d)[2d])) \\ &= H_M^{2d}(X \times F, d). \end{aligned}$$

We conclude using [36, Theorem 3.1(i) and Theorem 5.3.10].

To finish the proof, it remains to construct an isomorphism $\iota \circ h(F) \simeq \underline{Hom}(\iota \circ h(F), \mathbb{Q}(d)[2d])$. It suffices to do so in the stable triangulated category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Q})$ in which $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$ embeds fully faithfully by Voevodsky's cancellation theorem. (Recall that $\mathbf{DM}(k, \mathbb{Q})$ is defined as the homotopy category of $T = \mathbb{Q}_{tr}(\mathbb{A}^1/\mathbb{A}^1 - 0)$-spectra for the stable motivic model structure; for more details, see [5, Définition 2.5.27] in the special case where the valuation on $k$ is trivial.) In $\mathbf{DM}(k, \mathbb{Q})$, we have an isomorphism

$$\underline{Hom}(\iota \circ h(F), \mathbb{Q}(d)[2d]) \simeq \underline{Hom}(\iota \circ h(F), \mathbb{Q}(0)) \otimes \mathbb{Q}(d)[2d].$$

As the full embedding $\mathcal{M}_k \to \mathbf{DM}(k, \mathbb{Q})$ and the equivalence $\mathcal{M}_k \simeq \mathcal{M}_k^{DM}$ are tensorial, they preserve strong duals. From Lemma II.26, it follows that $\underline{Hom}(\iota \circ h(F), \mathbb{Q}(0))$ is canonically isomorphic to $\iota(F, \Delta_F, -d) = \iota \circ h(F) \otimes \mathbb{Q}(-d)[-2d]$. This gives the isomorphism $\underline{Hom}(\iota \circ h(F), \mathbb{Q}(d)[2d]) \simeq \iota \circ h(F)$ we want. $\square$

# CHAPTER III

# Motivic decomposition

## 3.1 Introduction

Recall that a *relative cellular variety* is a smooth and proper variety $X$ equipped with a finite increasing filtration by closed (not necessarily smooth) subvarieties

$$\phi = X_{-1} \subset X_0 \subset \dots X_n = X$$

such that each successive difference $X_{i/i-1} := X_i \setminus X_{i-1}$, called cell, admits a vector bundle $p_i : X_{i/i-1} \to Y_i$ to a smooth proper variety $Y_i$, called *base*. By a result of Karpenko [38, Corollary 6.11], the Chow motive of a relative cellular variety decomposes into the direct sum of the Chow motives of the bases suitably shifted and twisted. The shifts and twists depend on the ranks of $p_i$'s. The decomposition is given by the following strategy. First, one considers the graph of the vector bundle $p : GrX \to Y$ in $CH(GrX \times Y)$. Here $GrX := \coprod_{i=0}^{n} X_{i/i-1}$ and $Y := \coprod_{i=0}^{n} Y_i$. The closure of the graph is a cycle $\pi$ in $CH(X \times Y)$. The homotopy invariance property and functoriality of Chow groups and $K$-cohomology groups ([38, lemma 6.6, 6.7]) gives the following spilt exact sequence of Chow groups

$$0 \to CH(X_{i-1}) \to CH(X_i) \to CH(Y_i) \to 0. \qquad (3.1.1)$$

Using this we get an isomorphism $CH(Y) \cong CH(X)$. Then one shows that this isomorphism is given by ${}^t\pi \in CH(Y \times X)$.

We give the definition of a *Chow cellular* Deligne-Mumford stack in III.11. Lemma III.8 and corollary III.10 gives homotopy invariance property in our setting. The analogue of 3.1.1 is deduced from the Gysin triangle and proposition III.3. The motivic decomposition of *Chow cellular* Deligne-Mumford stack is given in III.13.

The existing proof of vanishing lemma of Voevodsky (cf . III.1) and the construction of the fully faithful embedding $\iota : \mathcal{M}_k^{eff} \to \mathbf{DM}^{eff}(k, \mathbb{Q})$ (cf. I.6) depends on resolution of singularities. Hence, throughout this chapter the base field $k$ will be of characteristic 0, unless otherwise mentioned. For perfect base field of positive characteristics $p$, we will get the same results for schemes as those mentioned in this chapter (with $\mathbb{Z}[1/p]$ coefficients), if the program mentioned in [40, section 1.6] is fully realized.

### 3.2 Voevodsky's vanishing lemma

The following lemma, which is essentially due to Voevodsky, is the main ingredient in our motivic decomposition theorem.

**Lemma III.1.** *Let $X, Y \in Sm/k$, such that $X$ is proper. Then*

$$Hom_{\mathbf{DM}^{eff}(k,\mathbb{Q})}(M(Y)(c)[2c], M(X)[1]) = 0.$$

*Proof.* The proof follows from [62, Corollary. 4.2.6] if $Y$ is proper. We follow the same argument here. Let $d = \dim(X)$. Since $X$ is proper, by [47, Example 20.11] we have

$$\underline{Hom}(M(X), \mathbb{Q}(d)[2d]) \cong M(X).$$

Hence, by [47, Proposition 14.16 and Theorem 19.3]

$$Hom_{\mathbf{DM}^{eff}(k,\mathbb{Q})}(M(Y)(c)[2c], M(X)[1]) = H_M^{2(d-c)+1}(Y \times X, d-c) = 0.$$

$\square$

**Remark III.2.** *Using the same argument of the previous lemma, one can deduce*

$$Hom_{\mathbf{DM}^{eff}(k,\mathbf{Z})}(M(Y)(c)[2c], M(X)[1]) = 0.$$

**Proposition III.3.** *Let $F$ be a smooth Deligne-Mumford stack and let $Z \subset F$ be a smooth and closed substack of codimension $c$. If $M(F \setminus Z)$ is a Chow motive, then there is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$,*

$$M(F) \cong M(Z)(c)[2c] \oplus M(F \setminus Z).$$

*Proof.* By [15, Lemma 3.9], there is an exact triangle,

$$M(F \setminus Z) \to M(F) \to M(Z)(c)[2c] \to M(F \setminus Z)[1].$$

We have to show that this triangle splits. Since $M(F \setminus Z)$ is a Chow motive and $M(Z)$ is a direct factor of $M(Y)$ for some smooth $k$-scheme $Y$ (cf. II.20), we have by Lemma III.1

$$Hom_{\mathbf{DM}^{eff}(k,\mathbb{Q})}(M(Z)(c)[2c], M(F \setminus Z)[1]) = 0.$$

This proves the existence of the required splitting. $\square$

**Remark III.4.** *We can explicitly construct the isomorphism*

$$M(F) \cong M(Z)(c)[2c] \oplus M(F \setminus Z)$$

*of Proposition III.3 using algebraic cycles. Let $X_i$ be a smooth and proper Deligne-Mumford stack of pure dimension $d_i$ for $1 \leq i \leq n$, and let $\sigma_i \in Ch^{c_i}((F \setminus Z) \times X_i)$ be a cycle of codimension $c_i$. Since each $X_i$ is proper, each cycle $\sigma_i$ induces a morphism*

$$\sigma_i : M(F \setminus Z) \to M(X_i)(c_i - d_i)[2(c_i - d_i)]$$

*in* $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$ *by remark I.30 and lemma II.23. Furthermore, assume that the morphism*

$$\cup_i \sigma_i : M(F \setminus Z) \to \bigoplus_{i=1}^{n} M(X_i)(c_i - d_i)[2(c_i - d_i)]$$

*is an isomorphism in* $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Q})$.

*If we can construct cycles* $\sigma_i' \in Ch^{c_i}(F \times X_i)$, *such that* $\sigma_i' \circ \iota = \sigma_i$, *then* $\iota : M(F \setminus Z) \to M(F)$ *splits to give an isomorphism as in Proposition III.3. The Zariski closures* $\sigma_i' := \overline{\sigma_i}$ *of* $\sigma_i$ *in* $F \times X_i$ *are such cycles and we get an isomorphism*

$$(\cup_i \sigma_i') \cup \sigma_Z : M(F) \to \left( \bigoplus_{i=1}^{n} M(X_i)(c_i - d_i)[2(c_i - d_i)] \right) \oplus M(Z)(c)[2c],$$

*where* $\sigma_Z \in Ch^{c+\dim(Z)}(F \times Z)$ *is the graph of the inclusion* $Z \subset F$.

**Remark III.5.** *If* $F$ *is a smooth scheme and* $Z$ *is a smooth closed subscheme of* $F$, *then the decomposition of III.3 holds in* $\mathbf{DM}^{eff}(k,\mathbb{Z})$.

### 3.3 Affine fibrations

**Definition III.6.** *A representable smooth morphism* $p : F' \to F$ *between Deligne-Mumford stacks is called a geometric affine fibration if the geometric fibers are affine spaces.*

In this section we prove that the motive of the total space of a geometric affine fibration is isomorphic to the motive of the base space.

**Lemma III.7.** *Let* $X$ *be a smooth separated irreducible* $k$*-scheme and let* $p : Y \to X$ *be a* geometric affine fibration. *Let* $n$ *be the relative dimension of* $p$. *Then* $X$ *can be filtered by open subschemes,*

$$\emptyset = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = X,$$

*such that for all $i$, $W_i := V_i \setminus V_{i-1}$ is smooth, and there is a pullback square of the form,*

$$\begin{tikzcd} \mathbf{A}^n_{U_i} \arrow[r] \arrow[d, "\acute{e}t"'] & U_i \arrow[d, "\acute{e}t"] \\ p^{-1}(W_i) \arrow[r] & W_i \end{tikzcd}$$

*such that the vertical morphisms are étale and surjective.*

*Proof.* Since $p$ is a *geometric affine fibration*, there exists an étale morphism $u_1 : U_1 \to X$ such that $Y \times_X U_1 \cong \mathbf{A}^n_{U_1}$. If $u_1$ is not surjective, let $V_1$ be the image of $u_1$ and let $Z_1 = X \setminus V_1$ be its complement. The scheme $Z_1$ is generically smooth as $char(k) = 0$ . It has a dense, smooth subscheme $W_2 \subset Z_1$ which is the image of an étale morphism $u_2 : U_2 \to Z_1$ such that $Y \times_X U_2 \cong \mathbf{A}^n_{U_2}$. Let $V_2 := U_1 \cup W_2$. Since $\mathrm{codim}_X(V_2 \setminus V_1) < \mathrm{codim}_X(V_1 \setminus V_0)$, this process terminates to give the required filtration of $X$. $\square$

**Lemma III.8.** *Let $X$ be a smooth separated $k$-scheme and let $p : Y \to X$ be a* geometric affine fibration*, Then $M(Y) \cong M(X)$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$.*

*Proof.* We can assume that $X$ is irreducible. Let $n$ be the relative dimension of $p$. By III.7, $X$ can be filtered by open subschemes,

$$\emptyset = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = X,$$

such that for all $i$, $W_i := V_i \setminus V_{i-1}$ is smooth, and there is a pullback square of the form,

$$\begin{tikzcd} \mathbf{A}^n_{U_i} \arrow[r] \arrow[d, "\text{ét}"'] & U_i \arrow[d, "\text{ét}"] \\ p^{-1}(W_i) \arrow[r] & W_i \end{tikzcd}$$

such that the vertical morphisms are étale and surjective.

We use induction on $i$ to show that $M(V_i) \cong M(p^{-1}(V_i))$ for all $i$. The case $i = 1$ follows from the observation that $M(V_1) \cong M((U_1)_\bullet)$ and $M(p^{-1}(V_1)) \cong M(p^{-1}(V_1) \times_{V_1} (U_1)_\bullet)$ where $(U_1)_\bullet$ is the Čech simplicial scheme associated to the étale cover $U_1 \to V_1$. In each simplicial degree, $p^{-1}(V_1) \times_{V_1} (U_1)_\bullet$ is isomorphic to $\mathbf{A}^n \times (U_1)_\bullet$. Hence the canonical morphism, $\mathbb{Q}_{tr}(p^{-1}(V_1) \times_{V_1} (U_1)_\bullet) \to \mathbb{Q}_{tr}((U_1)_\bullet)$, induces an $\mathbf{A}^1$-weak equivalence in each simplicial degree. This proves that $M(V_1) \cong M(p^{-1}(V_1))$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.

For the general case, we use the Gysin triangle from [62]. The required isomorphism then follows from the morphism of triangles,

$$\begin{tikzcd}
M(p^{-1}(V_{i-1})) \arrow[r] \arrow[d, "\wr"] & M(p^{-1}(V_i)) \arrow[r] \arrow[d] & M(p^{-1}(W_i))(c_i)[2c_i] \arrow[d, "\wr"] \\
M(V_{i-1}) \arrow[r] & M(V_i) \arrow[r] & M(W_i)(c_i)[2c_i]
\end{tikzcd}$$

where $c_i := \operatorname{codim}_{V_i} W_i$. □

**Remark III.9.** *Let $k$ be any perfect field. The morphism $p : Y \to X$ of the previous lemma also induces an isomorphism in $\mathbf{DM}^{eff}_{ét}(k, \mathbb{Z})$. If moreover the fibers of $p$ are affine spaces, then the isomorphism additionally holds in $\mathbf{DM}^{eff}(k, \mathbb{Z})$.*

**Corollary III.10.** *Let $p : F' \to F$ be a geometric affine fibration of smooth Deligne-Mumford stacks. Then $M(F') \cong M(F)$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.*

*Proof.* Let $U \to F$ be an atlas and let $V = U \times_F F' \to F'$ be the induced atlas for $F'$. Let $U_\bullet$ and $V_\bullet$ be the associated Čech simplicial schemes. There is a natural morphism $V_\bullet \to U_\bullet$. In each simplicial degree $i$, $V_i \to U_i$ is a geometric affine fibration. Therefore, the morphism $\mathbb{Q}_{tr}(V_\bullet) \to \mathbb{Q}_{tr}(U_\bullet)$ induces $\mathbf{A}^1$-weak equivalence in each simplicial degree. Hence $M(p) : M(F') \to M(F)$ is an isomorphism. □

### 3.4 Motivic decompositions

**Definition III.11.** *A* Chow cellular *Deligne-Mumford stack is a smooth Deligne-Mumford stack $F$ endowed with a finite increasing filtration by closed (not necessarily smooth) substacks,*

$$\emptyset = F_{-1} \subset F_0 \subset \cdots \subset F_n = F,$$

*such that the successive differences $F_{i\setminus i-1} = F_i \setminus F_{i-1}$, called* cells, *are smooth of pure codimension in $F$, and $M(F_{i\setminus i-1})$ are Chow motives for all $i$.*

*On the other hand, $F$ is said to be* relative geometrically cellular *if each cell $F_{i\setminus i-1}$ admits a geometric affine fibration $F_{i\setminus i-1} \to Y_i$ to a smooth, proper Deligne-Mumford stack $Y_i$, called the* base *of $F_{i\setminus i-1}$, then the stack. (Compare with [38, Definition 6.1]).*

**Example III.12.** *Let $X$ be a smooth and proper Deligne-Mumford stack over an algebraically closed field $k$ of characteristic zero, whose coarse moduli space is a scheme and let the an action of the multiplicative group $\mathbf{G}_m$ on $X$ is given. Then, $X$ admits a Białynicki-Birula decomposition [56, Theorem 3.5]. More precisely, if $F = \coprod_i F_i$ is the decomposition into connected components of the fixed point locus of the action of $\mathbf{G}_m$, then $X$ decomposes into a disjoint union of locally closed substacks $X_i$ which are $\mathbf{G}_m$-equivariant affine fibrations over the $F_i$'s.*

**Proposition III.13.** *Let $F$ be a Chow cellular Deligne-Mumford stack, retaining the notation of the definition III.11. Then there is an isomorphism in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q})$,*

$$M(F) \cong \bigoplus_{i=0}^{n} M(F_{i\setminus i-1})(c_i)[2c_i],$$

*where $c_i = \operatorname{codim}_F(F_{i\setminus i-1})$.*

*Proof.* We use descending induction on $i$ to show that

$$M(F \setminus F_i) \cong \bigoplus_{j=i+1}^{n} M(F_{j\setminus j-1})(c_j)[2c_j].$$

The result will then follow for $i = -1$.

For $i = n-1$, there is nothing to prove. Suppose we know the result for $i+1 \leq n-1$. As the motive $M(F_{i+1\setminus i})$ is Chow by assumption, Proposition III.3 gives an isomorphism

$$M(F \setminus F_i) \simeq M(F \setminus F_{i+1}) \oplus M(F_{i+1\setminus i})(c_i)[2c_i].$$

Hence we get our result. $\square$

Relative geometrically cellular Deligne-Mumford stacks are also Chow cellular. Indeed, by Corollary III.10, we have $M(F_{i\setminus i-1}) \cong M(Y_i)$ for all $i$ and the $M(Y_i)$'s are Chow motives by Theorem II.20. Hence, by proposition III.13, we get the following corollary.

**Corollary III.14.** *Let $F$ be a relative geometrically cellular stack, retaining the above notation. Then there is an isomorphism,*

$$M(F) \cong \bigoplus_{i=0}^{n} M(Y_i)(c_i)[2c_i],$$

*where $c_i = \mathrm{codim}_F(F_{i\setminus i-1})$.*

**Remark III.15.** *By Remark III.4 and the proof of Proposition III.13 it is clear that the isomorphism in Corollary III.14 is induced by the correspondences $\overline{\Gamma}_i \in Ch^{c_i+\dim(Y_i)}(F \times Y_i)$, where $\overline{\Gamma}_i \subset F \times Y_i$ is the closure of the graph of the morphism $F_{i\setminus i-1} \to Y_i$ inside $F \times Y_i$.*

**Remark III.16.** *Everything in this section works integrally in the classical case of relative cellular varieties. Hence, it yields a new proof of Karpenko's decomposition theorem [38, Corollary 6.11].*

**Example III.17.** *Let $X$ be a Deligne-Mumford stack as in example III.12. The motive of $X$ decomposes as follows*

$$M(X) \cong \bigoplus_i M(F_i)(c_i)[2c_i]$$

*where* $c_i = \operatorname{codim}_X(X_i)$.

## CHAPTER IV

# On a conjecture of Morel

In this chapter we study homotopy invariance property of the $\mathbb{A}^1$-connected component sheaf $a_{Nis}(\pi_0^{\mathbb{A}^1})$. The main results are Theorem IV.23, Theorem IV.2 and Corollary IV.3. The results of this chapter are essentially the results of [17].

## 4.1 Generalities on the Nisnevich local model structure

In this section we briefly recall the Nisnevich Brown-Gersten property and give some consequences on the $\pi_0$ functor.

Recall ([48, Definition 3.1.3]) that a cartesian square in $Sm/k$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X, \end{tikzcd}$$

is called an elementary distinguished square (in the Nisnevich topology), if $p$ is an étale morphism and $i$ is an open embedding such that $p^{-1}(X-U) \to (X-U)$ is an isomorphism (endowing these closed subsets with the reduced subscheme structure).

A space $\mathcal{X}$ is said to satisfy the Nisnevich Brown-Gersten property if for any elementary distinguished square in $Sm/k$ as above, the induced square of simplicial sets

$$\begin{tikzcd} \mathcal{X}(X) \arrow[r] \arrow[d] & \mathcal{X}(V) \arrow[d] \\ \mathcal{X}(U) \arrow[r] & \mathcal{X}(W) \end{tikzcd}$$

is homotopy cartesian (see [48, Definition 3.1.13]).

Any fibrant space for the Nisnevich local model structure satisfies the Nisnevich Brown-Gersten property ([48, Remark 3.1.15]).

A space is $\mathbb{A}^1$-fibrant if and only if it is fibrant in the local model structure and $\mathbb{A}^1$-local ([48, Proposition 2.3.19]).

There exist endofunctors $Ex$ (resp. $Ex_{\mathbb{A}^1}$) of $\triangle^{op}PSh(Sm/k)$ such that for any space $\mathcal{X}$, the object $Ex(\mathcal{X})$ is fibrant (resp. $Ex_{\mathbb{A}^1}\mathcal{X}$ is $\mathbb{A}^1$-fibrant). Moreover, there exists a natural morphism $\mathcal{X} \to Ex(\mathcal{X})$ (resp. $\mathcal{X} \to Ex_{\mathbb{A}^1}(\mathcal{X})$) which is a local weak equivalence (resp. $\mathbb{A}^1$-weak equivalence) ([48, Remark 3.2.5, Lemma 3.2.6, Theorem 2.1.66]).

**Remark IV.1.** *For the injective local model structure all spaces are cofibrant. Hence for any space $\mathcal{X}$ and for any $U \in Sm/k$,*

$$Hom_{\mathbf{H}_s(Sm/k)}(U, \mathcal{X}) = \pi_0(Ex(\mathcal{X})(U)).$$

*Since $Ex_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-local,*

$$Hom_{\mathbf{H}(k)}(U, \mathcal{X}) = Hom_{\mathbf{H}_s(Sm/k)}(U, Ex_{\mathbb{A}^1}(\mathcal{X})).$$

*Moreover $Ex_{\mathbb{A}^1}(\mathcal{X})$ is fibrant. Hence,*

$$Hom_{\mathbf{H}(k)}(U, \mathcal{X}) = \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X})(U)).$$

For any space $\mathcal{X}$, let $\pi_0(\mathcal{X})$ be the presheaf defined by

$$U \in Sm/k \mapsto Hom_{\mathbf{H}_s(Sm/k)}(U, \mathcal{X}).$$

**Theorem IV.2.** *Let $\mathcal{X}$ be a space. For any $X \in Sm/k$, such that $dim(X) \leq 1$, the canonical morphism*

$$\pi_0(\mathcal{X})(X) \to a_{Nis}(\pi_0(\mathcal{X}))(X)$$

*is surjective.*

Before giving the proof we note the following consequence.

**Corollary IV.3.** *For any space $\mathcal{X}$, the canonical morphism*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F)$$

*is bijective for all finitely generated separable field extensions $F/k$.*

*Proof.* For any $X \in Sm/k$,

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(X) = \pi_0(Ex_{\mathbb{A}^1}\mathcal{X})(X).$$

The canonical morphism

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F)$$

is surjective (applying theorem IV.2 for the space $Ex_{\mathbb{A}^1}(\mathcal{X})$). On the other hand, consider the following commutative diagram

$$\begin{tikzcd}
\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \arrow[r] \arrow[d] & \pi_0^{\mathbb{A}^1}(\mathcal{X})(F) \arrow[d, "\wr"] \\
a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F) \arrow[r] & a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(F),
\end{tikzcd}$$

where the horizontal morphisms are induced by the zero section $F \xrightarrow{s_0} \mathbb{A}^1_F$. The top horizontal morphism and the right vertical morphism are bijective. Hence the left vertical surjective morphism is injective. $\square$

The proof of theorem IV.2 depends on the relation between homotopy pullback of spaces and pullback of the presheaves of connected components of those spaces.

Let $I$ be a small category. There is a functor $(I/-) : I \to Cat$ such that for any $i \in I$, $(I/-)(i) = I/i$. Here $Cat$ is the category of small categories and $I/i$ is the over category. There is a functor $N : Cat \to \triangle^{op}Sets$, such that for any $J \in Cat$, the simplicial set $N(J)$ is the nerve of the category $J$. Define $N(I/-) := N \circ (I/-)$.

A set $S$ will be considered as a simplicial set in the obvious way : in every simplicial degree it is given by $S$ and faces and degeneracies are identities. These simplicial sets are called discrete simplicial sets.

**Lemma IV.4.** *Let $X : I \to \triangle^{op}Sets$ be a diagram of discrete simplicial sets. Then $lim_I X \cong holim_I X$.*

*Proof.* By adjointness ([14, Ch. XI 3.3])

$$Hom(\triangle^n \times N(I/-), X) = Hom(\triangle^n, holim_I X).$$

The fucntor $\pi_0 : (\triangle^{op}Sets)^I \to (Sets)^I$ is left adjoint to the fucntor $N : (Sets)^I \to (\triangle^{op}Sets)^I$, where $N$ maps a diagram of sets to the same diagram of discrete simplicial sets. Hence $Hom(\triangle^n \times N(I/-), X) = Hom(\bullet_I, X)$, where $\bullet_I$ is the diagram of sets given by the one element set for each $i \in I$. But $Hom(\bullet_I, X) = Hom(\bullet, lim_I X)$, by adjointness. Therefore, we get our result. □

**Remark IV.5.** *Let $X : I \to \triangle^{op}Sets$ be a diagram such that each $X(i)$ is fibrant for all $i \in I$. The canonical morphism $X(i) \to \pi_0(X(i))$ induces a morphism $holim_I(X) \to lim_I \pi_0(X)$. This gives the following morphism*

$$\pi_0(holim_I(X)) \to lim_I \pi_0(X). \tag{4.1.1}$$

**Lemma IV.6.** *Suppose that $I$ is the pullback category $1 \to 0 \leftarrow 2$ and let $D : I \to \triangle^{op}Sets$ be a digram $X \xrightarrow{p} Y \xleftarrow{q} Z$ such that $X, Y, Z$ are fibrant. Then the map (4.1.1) is surjective.*

*Proof.* By [14, Ch. XI 4.1.(iv), 5.6] $holim_I(X) \cong X' \times_Y Z$, where $X \to X' \xrightarrow{p'} Y$ is a factorisation of $p$ into a trivial cofibration followed by a fibration $p'$. Since $\pi_0(X) \cong \pi_0(X')$, it is enough to show that

$$\pi_0(X' \times_Y Z) \to \pi_0(X') \times_{\pi_0(Y)} \pi_0(Z)$$

is surjective. So we can assume that $p$ is a fibration. Let $s \in \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$. $s$ can be represented (not uniquely) by $(x, y, z)$, where $(x, z) \in X_0 \times Z_0$ and $y \in Y_1$ such that $d_0(y) = p(x)$ and $d_1(y) = q(z)$. Since $p$ is a fibration, we can lift the path $y$ to a path $y' \in X_1$ such that $d_0(y') = x$ and $x' := d_1(y')$ maps to $q(z)$. $holim_I D \cong X \times_Y Z$. Therefore $(x', z) \in holim_I D$ which maps to $s$. This proves the surjectivity. □

**Remark IV.7.** *Under the condition of lemma IV.6, the map (4.1.1) may not be injective. Indeed, if $Y$ is connected, $X$ is the universal cover of $Y$ and $Z = \bullet$, then (4.1.1) is injective if and only if $Y$ is simply connected.*

A noetherian $k$-scheme $X$, which is the inverse limit of a left filtering system $(X_\alpha)_\alpha$ with each transition morphism $X_\beta \to X_\alpha$ being an étale affine morphism between smooth $k$-schemes, is called an essentially smooth $k$-scheme. For any $X \in Sm/k$ and any $x \in X$, the local schemes $Spec(O_{X,x})$ and $Spec(O^h_{X,x})$ are essentially smooth $k$-schemes.

**Lemma IV.8.** *Let $\mathcal{X}$ be a space. For any essentially smooth discrete valuation ring $R$, the canonical morphism*

$$\pi_0(\mathcal{X})(R) \to a_{Nis}(\pi_0(\mathcal{X}))(R)$$

*is surjective.*

*Proof.* By remark IV.1 we can assume that $\mathcal{X}$ is fibrant.

Let $F = Frac(R)$ and let $R^h$ be the henselisation of $R$ at its maximal ideal. Suppose $s \in a_{Nis}(\pi_0(\mathcal{X}))(R)$. Then for the image of $s$ in $a_{Nis}(\pi_0(\mathcal{X}))(R^h)$, there exists a Nisnevich neighbourhood of the closed point $p : W \to Spec(R)$ and $s' \in \pi_0(\mathcal{X})(W)$, such that $s'$ gets mapped to $s|_W \in a_{Nis}(\pi_0(\mathcal{X}))(W)$. Let $L = Frac(W)$. For any finitely generated separable field extension $F/k$, the map $\pi_0(\mathcal{X})(F) \to a_{Nis}(\pi_0(\mathcal{X}))(F)$ is bijective. Hence, $s'|_L$ is same as $s|_L$. We get two sections $s' \in \pi_0(\mathcal{X})(W)$ and $s|_F \in \pi_0(\mathcal{X})(F)$, such that $s'|_L = s|_L$. By lemma IV.6 and the fact that $\mathcal{X}$ satisfies the Nisnevich Brown-Gersten property, we find an element $s_v \in \pi_0(\mathcal{X})(R)$ which gets mapped to $s$. Therefore, $\pi_0(\mathcal{X})(R) \to a_{Nis}(\pi_0(\mathcal{X}))(R)$ is surjective. □

*Proof of theorem IV.2.* Let $X \in Sm/k$ and $dim(X) = 1$. Let $\alpha$ be an element of $a_{Nis}(\pi_0(\mathcal{X}))(X)$. This $\alpha$ gives $\alpha_p \in a_{Nis}(\pi_0(\mathcal{X}))(O_{X,p})$ for every codimension 1 point $p \in X$, such that $\alpha_p|_{K(X)} = \alpha_q|_{K(X)}$, for all $p, q \in X^{(1)}$. By the surjectivity of

$$\pi_0(\mathcal{X})(O_{X,p}) \to a_{Nis}(\pi_0(\mathcal{X}))(O_{X,p})$$

and bijectivity of

$$\pi_0(\mathcal{X})(K(X)) \to a_{Nis}(\pi_0(\mathcal{X}))(K(X)),$$

we get elements $\alpha'_p \in \pi_0(\mathcal{X})(O_{X,p})$ mapping to $\alpha_p$, such that $\alpha'_p|_{K(X)} = \alpha'_q|_{K(X)}$ for $p, q \in X^{(1)}$.

Fix a $p \in X^{(1)}$. There exists an open set $U$ and $\beta \in \pi_0(\mathcal{X})(U)$, such that $\beta|_{O_{X,p}} = \alpha'_p$. Let $\beta' \in a_{Nis}(\pi_0(\mathcal{X})(U))$ be the image of $\beta$. Suppose that $\beta' \neq \alpha|_U$, but $\beta'|_{O_{X,p}} = \alpha_p$. Hence there exsists $U' \subset U$, such that $\beta'|_{U'} = \alpha|_{U'}$.

So we can assume by Noetherian property of $X$ that there exists a maximal open set $U \subset X$ and $\alpha' \in \pi_0(\mathcal{X})(U)$, such that $\alpha'$ gets mapped to $\alpha|_U$. If $U \neq X$, then there exists a codimension one point $q \in X \setminus U$. We can get an open neighborhood $U_q$ and an element $\alpha'' \in \pi_0(\mathcal{X})(U_q)$, such that $\alpha''$ gets mapped to $\alpha|_{U_q}$. But by construction of these $\alpha'', \alpha'$ we know that $\alpha''|_{K(X)} = \alpha'|_{K(X)}$. Hence there exists an open set $U' \subset U_q \cap U$, such that $\alpha''|_{U'} = \alpha'|_{U'}$. Let $Z = U_q \cap U \setminus U'$. Since $dim(X) = 1$, the set $Z$ is finite collection of closed points. Therefore, $Z$ is closed in $U$. Let $U'' = U \setminus Z$ be the open subset of $U$. Note that $U'' \cap U_q = U'$. Denote $U'' \cup U_q = U \cup U_q$ by $V$.

Let $\alpha'|_{U''} \in \pi_0(\mathcal{X})(U'')$ be the restriction of $\alpha'$ to $U''$. Hence, $\alpha'|_{U''}$ gets mapped to $\alpha|_{U''}$ and $\alpha'|_{U''}$ restricted to $U'$ is same as $\alpha''$ restricted to $U'$. As $\mathcal{X}$ is Nisnevich fibrant, it satisfies the Zariski Brown-Gersten property. By lemma IV.6, we get a section $s_V \in \pi_0(\mathcal{X})(V)$ which gets mapped to $s|_V$. This gives a contradiction to the maximality of $U$. This finishes the proof of the theorem. $\square$

## 4.2 $H$-groups and homogeneous spaces

In this section we prove $\mathbb{A}^1$-invariance of $a_{Nis}(\pi_0^{\mathbb{A}^1})$ for $H$-groups and homogeneous spaces for $H$-groups.

**Definition IV.9.** *Let $\mathcal{X}$ be a pointed space, i.e., $\mathcal{X}$ is a space endowed with a morphism $x : Spec(k) \to \mathcal{X}$. It is called an $H$-space if there exists a base point preserving morphism $\mu : (\mathcal{X} \times \mathcal{X}) \to \mathcal{X}$, such that $\mu \circ (x \times id_{\mathcal{X}})$ and $\mu \circ (id_{\mathcal{X}} \times x)$ are equal to $id_{\mathcal{X}}$ in $\mathbf{H}(k)$. Here $\mathcal{X} \times \mathcal{X}$ is pointed by $(x, x)$. It is called an $H$-group if :*

1. *$\mu \circ (\mu \times id_{\mathcal{X}})$ is equal to $\mu \circ (id_{\mathcal{X}} \times \mu)$ in $\mathbf{H}(k)$ modulo the canonical isomorphism $\alpha : \mathcal{X} \times (\mathcal{X} \times \mathcal{X}) \to (\mathcal{X} \times \mathcal{X}) \times \mathcal{X}$.*

2. *There exists a morphism $(.)^* : \mathcal{X} \to \mathcal{X}$, such that $\mu \circ (id_{\mathcal{X}}, (.)^*)$ and $\mu \circ ((.)^*, id_{\mathcal{X}})$*

*are equal to the constant map $c : \mathcal{X} \to \mathcal{X}$ in $\mathbf{H}(k)$. Here the image of the constant map $c$ is $x$.*

**Remark IV.10.** *Recall from [48, 3.2.1] that*

$$Ex_{\mathbb{A}^1} = Ex^{\mathcal{G}} \circ (Ex^{\mathcal{G}} \circ Sing_*^{\mathbb{A}^1})^{\mathbb{N}} \circ Ex^{\mathcal{G}}.$$

*The fucntors $Ex^{\mathcal{G}}$ and $Sing_*^{\mathbb{A}^1}$ commutes with finite limits by [48, 2.3.2, Theorem 2.1.66]. Also filtered colimit commutes with finite products. Therefore, $Ex_{\mathbb{A}^1}$ commutes with finite products. If $\mathcal{X}$ is an $H$-group as described in IV.9, then the morphisms $Ex_{\mathbb{A}^1}(x)$, $Ex_{\mathbb{A}^1}(\mu)$ and $Ex_{\mathbb{A}^1}((.)^*)$ satisfy the conditions of the definition IV.9. Hence, $Ex_{\mathbb{A}^1}(\mathcal{X})$ is also an $H$-group.*

*Suppose that $a, b, c \in \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))(U)$ for some $U \in Sm/k$. Let $f, g : \mathcal{Y} \to \mathcal{Z}$ be morphisms between $\mathbb{A}^1$-fibrant spaces such that $f$ is equal to $g$ in $\mathbf{H}(k)$, then $f$ and $g$ are simplicially homotopic. Using this, we get $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$, $\mu(a, x) = a = \mu(x, a)$ and $\mu(a, a^*) = \mu(a^*, a) = x$. Hence, $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is a presheaf of groups.*

Let $\mathcal{X}$ be an $H$-group. Let $\mathcal{Y}$ be a space.

**Definition IV.11.** *The space $\mathcal{Y}$ is called an $\mathcal{X}$-space if there exists a morphism $a : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$, such that the following diagram*

$$\begin{tikzcd}
\mathcal{X} \times (\mathcal{X} \times \mathcal{Y}) \arrow[r, "id_{\mathcal{X}} \times a"] \arrow[d, "a_{\mathcal{X}} \times id_{\mathcal{Y}}"] & \mathcal{X} \times \mathcal{Y} \arrow[d, "a"] \\
\mathcal{X} \times \mathcal{Y} \arrow[r, "a"] & \mathcal{Y}
\end{tikzcd}$$

*commutes in $\mathbf{H}(k)$.*

**Definition IV.12.** *Let $\mathcal{X}$ be an $H$-group and let $\mathcal{Y}$ be an $\mathcal{X}$-space. $\mathcal{Y}$ is called a homogeneous $\mathcal{X}$-space if for any essentially smooth henselian $R$, the presheaf of groups $\pi_0^{\mathbb{A}^1}(\mathcal{X})(R)$ acts transitively on $\pi_0^{\mathbb{A}^1}(\mathcal{Y})(R)$.*

**Remark IV.13.** *$\mathcal{Y}$ is a homogeneous $\mathcal{X}$-space if and only if the colimit of the diagram $\pi_0^{\mathbb{A}^1}(\mathcal{Y}) \xleftarrow{pr} \pi_0^{\mathbb{A}^1}(\mathcal{X}) \times \pi_0^{\mathbb{A}^1}(\mathcal{Y}) \xrightarrow{a} \pi_0^{\mathbb{A}^1}(\mathcal{Y})$ is Nisnevich locally trivial.*

**Lemma IV.14.** *If*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & D \end{tikzcd}$$

*is a homotopy cocartesian square of spaces then, after applying $a_{Nis}(\pi_0)$, one gets a cocartesian square of sheaves.*

*Proof.* Let $S \in PSh(Sm/k)$ and let $\iota(S)$ be the simplicial presheaf such that in every simplicial degree $k$, $\iota(S)_k = S$. The face and degeneracy morphisms are identity morphisms. This gives a functor $\iota : PSh(Sm/k) \to \triangle^{op}PSh(T)$ which is right adjoint to $\pi_0$. Hence $a_{Nis}(\pi_0)$ also has a right adjoint $\iota : Sh(Sm/k) \to \triangle^{op}Sh(Sm/k)$. This implies, $a_{Nis}(\pi_0)$ commutes with colimits. Let $B \xrightarrow{f} A' \xrightarrow{g} A$ be a factorisation of $B \to A$, such that $f$ is a cofibration and $g$ is a trivial fibration. Homotopy colimit of the digram $A \leftarrow B \to C$ is weakly equivalent to the colimit of $A' \leftarrow B \to C$. As $a_{Nis}(\pi_0)$ commutes with colimits and $a_{Nis}(\pi_0(A)) \cong a_{Nis}(\pi_0(A'))$, we get our result. □

**Corollary IV.15.** *Let $\mathcal{Y}$ be an $\mathcal{X}$-space. $\mathcal{Y}$ is a homogeneous $\mathcal{X}$-space if and only if the homotopy pushout of $Ex_{\mathbb{A}^1}(\mathcal{Y}) \xleftarrow{pr} Ex_{\mathbb{A}^1}(\mathcal{X}) \times Ex_{\mathbb{A}^1}(\mathcal{Y}) \xrightarrow{a} Ex_{\mathbb{A}^1}(\mathcal{Y})$ is connected.*

*Proof.* The proof follows from lemma IV.14 and remark IV.13. □

**Lemma IV.16.** *Let $\mathcal{Y}$ be an $\mathcal{X}$-space. $\mathcal{Y}$ is a homogeneous $\mathcal{X}$-space if the homotopy pushout of $\mathcal{Y} \xleftarrow{pr} \mathcal{X} \times \mathcal{Y} \xrightarrow{a} \mathcal{Y}$ is connected.*

*Proof.* By [48, corolarry 2.3.22], the canonical morphism $a_{Nis}(\pi_0(\mathcal{X})) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ (resp. $a_{Nis}(\pi_0(\mathcal{Y})) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$ is surjective as morphism of Nisnevich sheaves. Hence, Nisnevich locally the action of $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ on $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$ is transitive. $\square$

**Lemma IV.17.** *Let $G, G'$ be groups acting on pointed sets $S, S'$ by action maps $a, a'$ respectively. Suppose that $f : G \to G'$ is a group homorphism and let $s : S \to S'$ be a morphism of pointed sets with trivial kernel such that $s \circ a = a' \circ (f \times s)$. If $G$ acts transitively on $S$, then $s$ is injective.*

*Proof.* Let $b_S$ (resp. $b_{S'}$) be the base point of $S$ (resp. $S'$) and let $a, b \in S$. Since $G$ acts transitively on $S$, there exist $g, g' \in G$ such that $a(g, b_S) = a$ and $a(g', b_S) = b$. If $s(a) = s(b)$, then $a'(f(g), b_{S'}) = a'(f(g'), b_{S'})$. Hence $a'(f(g^{-1}.g'), b_{S'}) = b_{S'}$. So $s(a(g^{-1}.g', b_S)) = b_{S'}$. But $s$ is a morphism of pointed sets with trivial kernel, therefore $a(g^{-1}.g', b_S) = b_S$. This implies $a = a(g, b_S) = a(g', b_S) = b$. $\square$

Let $\tilde{Sm}/k$ be the category whose objects are same as objects of $Sm/k$, but the morphisms are smooth morphisms. The following argument is taken from [49, Corollary 5.9]

**Lemma IV.18.** *Let $S$ be a Nisnevich sheaf on $Sm/k$. Suppose that for all essentially smooth henselian $X$, the map $S(X) \to S(K(X))$ is injective. Then $S(Y) \to S(K(Y))$ is injective, for all connected $Y \in Sm/k$.*

*Proof.* Let $S'$ be the presheaf on $\tilde{Sm}/k$, given by

$$X \in \tilde{Sm}/k \mapsto \prod_i S(K(X_i)),$$

where $X_i$'s are the connected components of $X$. Then $S'$ is a Nisnevich sheaf on $\tilde{Sm}/k$ (as every Nisnevich covering of some $X \in \tilde{Sm}/k$ splits over some open dense $U \subset X$). The canonical morphism $S \to S'$ is injective on Nisnevich stalks. Hence $S \to S'$ is sectionwise injective. $\square$

**Corollary IV.19.** *Let $S$ be a Nisnevich sheaf on $Sm/k$. Suppose that for all essentially smooth henselian $X$, the map $S(X) \to S(K(X))$ is injective. Then $S(Y) \to S(U)$ is injective for any $Y \in Sm/k$ and any open dense $U \subset Y$.*

*Proof.* We can assume that $Y$ is connected. By lemma IV.18, the morphism $S(Y) \to S(K(Y))$ is injective and $S(U) \to S(K(Y))$ is injective, hence $S(Y) \to S(U)$ is injective. $\square$

**Lemma IV.20.** *Let $S$ be a Zariski sheaf on $Sm/k$, such that $S(X) \to S(U)$ is injective for any $X \in Sm/k$ and for any open dense $U \subset X$. Then $S$ is $\mathbb{A}^1$-invariant if and only if $S(F) \to S(\mathbb{A}^1_F)$ is bijective for every finitely generated separable field extension $F/k$.*

*Proof.* The only if part is clear. We need to show that for any connected $X \in Sm/k$, the morphism $S(\mathbb{A}^1_X) \to S(X)$ (induced by the zero section), is bijective. Let $F := K(X)$. In the following commutative diagram

$$\begin{array}{ccc} S(\mathbb{A}^1_X) & \longrightarrow & S(X) \\ \downarrow & & \downarrow \\ S(\mathbb{A}^1_F) & \longrightarrow & S(F) \end{array}$$

the left vertical, the right vertical and the bottom horizontal morphisms are injective, thus the top horizontal surjective morphism is injective. $\square$

We recall the following from [30] and [49, Corollary 5.7]

**Theorem IV.21.** *Let $X$ be a smooth (or essentially smooth) $k$-scheme, $s \in X$ be a point and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open subscheme $\Omega \subset X$ containing $s$ and a closed subscheme $Z' \subset \Omega$, of codimension $d - 1$, containing $Z_\Omega := Z \cap \Omega$ and such that for any $n \in \mathbb{N}$ and for any $\mathbb{A}^1$-fibrant* *space $\mathcal{X}$, the map*

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \to \pi_n(\mathcal{X}(\Omega/(\Omega - Z')))$$

*is the trivial map. In particular, if $Z$ has codimension 1 and $X$ is irreducible, $Z'$ must be $\Omega$. Thus for any $n \in \mathbb{N}$ the map*

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \to \pi_n(\mathcal{X}(\Omega))$$

*is the trivial map.*

**Remark IV.22** ([49])**.** *Let $X$ be an essentially smooth local ring and let $x$ be the closed point. Let $U \subset X$ be an open set. We have the following exact sequence of sets and groups for any $\mathbb{A}^1$-fibrant space $\mathcal{X}$ :*

$$\cdots \to \pi_1(\mathcal{X})(X) \to \pi_1(\mathcal{X})(U) \to \pi_0(\mathcal{X})(X/U) \to \pi_0(\mathcal{X})(X) \to \pi_0(\mathcal{X})(U)$$

*Applying theorem IV.21 to $X$ and its closed point $x$, we see that $\Omega = X$ and the morphisms*

$$\pi_n(\mathcal{X})(X/U) \to \pi_n(\mathcal{X})(X)$$

*are trivial. Hence the morphism of pointed sets*

$$\pi_0(\mathcal{X})(X) \to \pi_0(\mathcal{X})(U)$$

*has trivial kernel. Taking colimit over open sets, this gives the morphism of pointed sets*

$$\pi_0(\mathcal{X})(X) \to \pi_0(\mathcal{X})(K(X))$$

*which has trivial kernel. In particular if $X$ is henselian, then the morphism of pointed sets*

$$a_{Nis}(\pi_0(\mathcal{X}))(X) \to a_{Nis}(\pi_0(\mathcal{X}))(K(X))$$

*has trivial kernel.*

**Theorem IV.23.** *Let $\mathcal{X}$ be an $H$-group and $\mathcal{Y}$ be a homogeneous $\mathcal{X}$-space. Then $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ and $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$ are $\mathbb{A}^1$-invariant.*

*Proof.* For any connected $X \in Sm/k$ and any $x \in X$, the morphisms of pointed sets

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O^h_{X,x}) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(O^h_{X,x}))$$

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(O^h_{X,x}) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(O^h_{X,x}))$$

have trivial kernel by remark IV.22. By lemma IV.17 and the fact that $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O^h_{X,x})$ is a group, the morphisms mentioned above are injective morphisms of sets. By lemma IV.18, for every $X \in Sm/k$, the morphisms

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(X))$$

and

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(X))$$

are injective. Hence for any $X \in Sm/k$ and any open dense subscheme $U \subset X$, the morphisms

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U)$$

and

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(U)$$

are injective by corollary IV.19,. Now applying corollary IV.3 and lemma IV.20, we get our result. □

**Remark IV.24.** *If $\mathcal{X}$ is an $H$-group, then*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \to a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R)$$

*is bijective for any essentially smooth discrete valuation ring $R$. Indeed, using remark IV.22 one can easily show that for any essentially smooth discrete valuation ring $R$, the group homomorphism*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \to \pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R))$$

*is injective. On the other hand, consider the following commutative diagram*

$$\begin{tikzcd} \pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \arrow[r] \arrow[d] & \pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R)) \arrow[d, "\wr"] \\ a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R) \arrow[r] & a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R)) \end{tikzcd}$$

*where the bottom horizontal morphism is injective by theorem IV.23. The right vertical injective morphism is surjective by lemma IV.8. Hence it is bijective.*

### 4.3 Application and comments

By gathering known facts from [64], [50, Theorem 2.4] and [23, Corollary 5.10] one can show that for any connected linear algebraic group $G$, such that the almost simple factors of the universal covering (in algebraic group theory sense) of the semisimple part of $G$ is isotropic and retract $k$-rational ([23, Definition 2.2]), the sheaf $a_{Nis}(\pi_0^{\mathbb{A}^1}(G))$ is $\mathbb{A}^1$-invariant. By IV.23, we have the following generalisation.

**Corollary IV.25.** *Let $G$ be any sheaf of groups on $Sm/k$ and $B$ be any subsheaf of groups. Then $a_{Nis}(\pi_0^{\mathbb{A}^1}(G))$ is $\mathbb{A}^1$-invariant and $a_{Nis}(\pi_0^{\mathbb{A}^1}(G/B))$ is $\mathbb{A}^1$-invariant. Here $G/B$ is the quotient sheaf in Nisnevich topology.*

We recall from [49, Definition 7] the following definition.

**Definition IV.26.** *A sheaf of groups $G$ on $Sm/k$ is called strongly $\mathbb{A}^1$-invariant if for any $X \in Sm/k$, the map*

$$H^i_{Nis}(X, G) \to H^i_{Nis}(\mathbb{A}^1_X, G)$$

*induced by the projection* $\mathbb{A}^1_X \to X$*, is bijective for* $i \in \{0,1\}$.

Let $\mathcal{X}$ be a pointed space. By [49, Theorem 9], for any pointed simplicial persheaf $\mathcal{X}$, the sheaf of groups $a_{Nis}(\pi_0(\Omega(Ex_{\mathbb{A}^1}(\mathcal{X})))) = \pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant. Here $x$ is the base point of $\mathcal{X}$ and $\Omega(Ex_{\mathbb{A}^1}(\mathcal{X}))$ is the loop space of $Ex_{\mathbb{A}^1}(\mathcal{X})$. So for any space $\mathcal{X}$, which is the loop space of some $\mathbb{A}^1$-local space $\mathcal{Y}$, [49, Thoerem 9] gives the $\mathbb{A}^1$-invariance property for $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$. We end this section by showing that there exists an $\mathbb{A}^1$-local $H$-group which is not a loop space of some $\mathbb{A}^1$-local space. This will imply that the statement of the theorem IV.23 for $H$-groups is not a direct consequence of [49, Theorem 9]. It is enough to show that there exists sheaf of groups $G$ which is $\mathbb{A}^1$-invariant, but not strongly $\mathbb{A}^1$-invariant.

Let $\mathbb{Z}[\mathbb{G}_m]$ be the free presheaf of abelian groups generated by $\mathbb{G}_m$.

**Remark IV.27.** *For any* $X \in Sm/k$ *and a dominant morphism* $U \to X$*, the canonical morphism* $\mathbb{Z}[\mathbb{G}_m](X) \to \mathbb{Z}[\mathbb{G}_m](U)$ *is injective. Indeed, any nonzero* $a \in \mathbb{Z}[\mathbb{G}_m](X)$ *can be written as* $a = \sum_{i=1}^n a_i.g_i$*, where* $g_i \in \mathbb{G}_m(X)$ *and* $a_i \in \mathbb{Z} \setminus \{0\}$ *such that* $g_i \neq g_{i'}$ *for* $i \neq i'$*. Suppose* $a|_U = 0$*, i.e.,* $\sum_{i=1}^n a_i.g_i|_U = 0$*. Since* $\mathbb{G}_m(X) \to \mathbb{G}_m(U)$ *is injective,* $g_i|_U \neq g_{i'}|_U$ *for* $i \neq i'$*. This implies* $a_i = 0$ *for all* $i$*. Hence* $a = 0$.

The presheaf $\mathbb{Z}[\mathbb{G}_m]$ is not a Nisnevich sheaf. But it is not far from being a Nisnevich sheaf.

**Lemma IV.28.** *The Nisnevich sheafification* $a_{Nis}(\mathbb{Z}[\mathbb{G}_m])$ *is the presheaf that associates to every smooth* $k$*-scheme* $X = \coprod_i X_i$*, the abelian group* $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$*, where* $X_i$*'s are the connected components of* $X$*.*

*Proof.* Let $\mathcal{F}$ be the presheaf that associates to every smooth $k$-scheme $X = \coprod_i X_i$, the abelian group $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$, where $X_i$'s are the connected components of $X$.

It is enough to prove that $\mathcal{F}$ is a Nisnevich sheaf. We need to show that for any elementary distinguished square in $Sm/k$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X, \end{tikzcd}$$

the induced commutative square

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[d] \\ \mathcal{F}(U) \arrow[r] & \mathcal{F}(W) \end{tikzcd}$$

is cartesian. By the construction of $\mathcal{F}$ we can assume that $X, W, V, U$ are connected. So, it is enough to prove that

$$\begin{tikzcd} \mathbb{Z}[\mathbb{G}_m](X) \arrow[r] \arrow[d] & \mathbb{Z}[\mathbb{G}_m](V) \arrow[d] \\ \mathbb{Z}[\mathbb{G}_m](U) \arrow[r] & \mathbb{Z}[\mathbb{G}_m](W) \end{tikzcd}$$

is cartesian. Let $a \in \mathbb{Z}[\mathbb{G}_m](U)$ and let $b \in \mathbb{Z}[\mathbb{G}_m](V)$ such that $a|_W = b|_W$. We can write $a = \sum_{i=1}^n a_i.f_i$ and $b = \sum_{j=1}^m b_j.g_j$, where $a_i, b_j \in \mathbb{Z} \setminus \{0\}$ and $(f_i, g_j) \in \mathbb{G}_m(U) \times \mathbb{G}_m(V)$ such that $f_i \neq f_{i'}$ and $g_j \neq g_{j'}$ for all $i \neq i'$ and $j \neq j'$. Since all the morphisms are dominant, $g_j|_W \neq g_{j'}|_W$ and $f_i|_W \neq f_{i'}|_W$ for all $i \neq i'$ and $j \neq j'$. Hence, for every $i$ there exists atmost one $j$ such that $f_i|_W = g_j|_W$. Suppose for some $f_{i'}$, $f_{i'}|_W \neq g_j|_W$ for all $j$. Then we can write

$$(\sum_{i=1}^n a_i.f_i|_W) - (\sum_{j=1}^m b_j.g_j|_W) = a_{i'}f_{i'} + \sum_{k=1}^l c_k.h_k = 0,$$

where $h_k \neq h_{k'}$ for all $k \neq k'$ and $f_{i'} \neq h_k$ for all $k$. This implies $a_{i'} = 0$, which gives a contradiction. Hence, for every $i$ there exists exactly one $j$ such that $f_i|_W = g_j|_W$. Therefore, $m = n$. Also we can write $a = \sum_{i=1}^n a'_i.f'_i$, such that $a'_i = b_i$ and $f'_i|_W =$

$g_i|_W$. Since $\mathbb{G}_m$ is a Nisnevich sheaf, we get $g'_i \in \mathbb{G}_m(X)$ which restricts to $f'_i$ and $g_i$. This gives a section $c = \sum_{i=1}^{n} b_i.g'_i \in \mathbb{Z}[\mathbb{G}_m](X)$ which restricts to $a$ and $b$. The uniqueness of $c$ follows from the remark IV.27. □

As $\mathbb{G}_m$ is pointed by 1, $a_{Nis}(\mathbb{Z}[\mathbb{G}_m]) \cong \mathbb{Z} \oplus \mathbb{Z}(\mathbb{G}_m)$. Here $\mathbb{Z}$ is the sheaf generated by the point 1. Let $A$ be a sheaf of abelian groups on $Sm/k$. To give a morphism $\mathbb{G}_m \to A$, such that 1 gets mapped to $0 \in A$, is equivalent to give a morphism $\mathbb{Z}(\mathbb{G}_m) \to A$ of abelian sheaves. Since $\mathbb{G}_m$ is $\mathbb{A}^1$-invariant, $a_{Nis}(\mathbb{Z}[\mathbb{G}_m])$ is $\mathbb{A}^1$-invariant. This implies $\mathbb{Z}(\mathbb{G}_m)$ is $\mathbb{A}^1$-invariant.

**Remark IV.29.** *Let $\sigma_1 : \mathbb{G}_m \to \underline{K}_1^{MW}$ be the canonical pointed morphism (see [49, page 86]). For any finitely generated separable field extension $F/k$, the morphism maps $u \in F^*$ to the corresponding symbol $[u] \in K_1^{MW}(F)$. The induced morphism $\mathbb{Z}(\mathbb{G}_m) \to \underline{K}_1^{MW}$ is not injective. Indeed, we can choose $u \in F^* \setminus 1$ such that $u(u-1)$ is not 1. The element $[u(u-1)] - [u] - [u-1]$ is zero in $K_1^{MW}(F)$, but it is non zero in $\mathbb{Z}(\mathbb{G}_m)(F)$.*

**Lemma IV.30.** *The $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathbb{Z}(\mathbb{G}_m)$ is not strongly $\mathbb{A}^1$-invariant.*

*Proof.* Suppose $\mathbb{Z}(\mathbb{G}_m)$ is strongly $\mathbb{A}^1$-invaraint. Then by [49, Theorem 2.37], the morphism $id : \mathbb{Z}(\mathbb{G}_m) \to \mathbb{Z}(\mathbb{G}_m)$ can be written as $\phi \circ \sigma_1$ for some unique $\phi$. This implies $\sigma_1$ is injective which contradicts remark IV.29. □

# CHAPTER V

## 5.1 Appendix A

As usual, we fix a base field $k$ of characteristic 0. (Varieties will be always defined over $k$.) Recall that $\mathcal{M}_k^{\mathit{eff}}$ is the category of effective Chow motives with rational coefficients. We will have to consider the following categories of varieties.

1. $\mathcal{V}_k$: the category of smooth and projective varieties.
2. $\mathcal{V}'_k$: the category of projective varieties having at most global quotient singularities, i.e., those that can be written as a quotient of an object of $\mathcal{V}_k$ by a finite group.
3. $\mathcal{N}_k$: the category of projective normal varieties.
4. $\mathcal{P}_k$: the category of all projective varieties.

We have the chain of inclusions

$$\mathcal{V}_k \subset \mathcal{V}'_k \subset \mathcal{N}_k \subset \mathcal{P}_k.$$

Given $N \in \mathcal{M}_k^{\mathit{eff}}$ we define a functor $\omega_N : \mathcal{V}_k^{op} \to Vec_{\mathbb{Q}}$ by

$$\omega_N(X) = Hom_{\mathcal{M}_k^{\mathit{eff}}}(M(X), N), \text{ for } X \in \mathcal{V}_k.$$

We thus have a functor $\omega : \mathcal{M}_k^{\mathit{eff}} \to PSh(\mathcal{V}_k)$ given by $N \mapsto \omega_N$.

**Theorem V.1.** *The functor $\omega : \mathcal{M}_k^{\mathit{eff}} \to PSh(\mathcal{V}_k)$ is fully faithful, i.e., for every $M, N \in \mathcal{M}_k^{\mathit{eff}}$, the natural morphism*

$$Hom_{\mathcal{M}_k^{\mathit{eff}}}(M, N) \to Hom(\omega_M, \omega_N) \tag{5.1.1}$$

*is bijective.*

**Remark V.2.** *The statement of the theorem appears without proof in [55, 2.2] and is also mentioned in [58, p. 12].*

**Lemma V.3.** *The functor $\omega$ is faithful.*

*Proof.* To show that the map (5.1.1) is injective, we may assume that $M = M(X)$ and $N = M(Y)$ for $X, Y \in \mathcal{V}_k$. In this case, (5.1.1) has a retraction given by $\alpha \in Hom(\omega_M, \omega_N) \mapsto \alpha(id_X)$. Hence it is injective. □

**Definition V.4.**

1. *The pcdh topology on $\mathcal{P}_k$ is the Grothendieck topology generated by the covering families of the form $(X' \xrightarrow{p_{X'}} X, Z \xrightarrow{p_Z} X)$ such that $p_{X'}$ is a proper morphism, $p_Z$ is a closed embedding and $p_{X'}^{-1}(X - p_Z(Z)) \to X - p_Z(Z)$ is an isomorphism. To avoid problems, we also add the empty family to the covers of the empty scheme.*

2. *The $fh$ topology on $\mathcal{N}_k$ is the topology associated to the pretopology formed by the finite families $(f_i : Y_i \to X)_{i \in I}$ such that $\cup_i f_i : \coprod_{i \in I} Y_i \to X$ is finite and surjective.*

**Lemma V.5.** *Let $M \in \mathcal{M}_k^{\mathit{eff}}$. The presheaf $\omega_M$ can be extended to a presheaf $\omega'_M$ on $\mathcal{V}'_k$ such that for $X = X'/G$ with $X' \in \mathcal{V}_k$ and $G$ a finite group, we have $\omega'_M(X) = \omega_M(X')^G$.*

*Proof.* By [21, Example 8.3.12], we can define refined intersection class with rational coefficients which can be used to define a category of effective Chow motives $\mathcal{M}_k'^{eff}$. Moreover, the canonical functor $\phi : \mathcal{M}_k^{eff} \to \mathcal{M}_k'^{eff}$, induced by the inclusion $\mathcal{V}_k \to \mathcal{V}_k'$, is an equivalence of categories (cf. [9, Proposition 1.2]). For $X \in \mathcal{V}_k'$, we set

$$\omega_M'(X) = Hom_{\mathcal{M}_k'^{eff}}(M(X), \phi(M)).$$

In this way we get a presheaf $\omega_M'$ on $\mathcal{V}_k'$ which extends the presheaf $\omega_M$. Moreover, the identification $\omega_M'(X'/G) = \omega_M(X')^G$ is clear. □

**Lemma V.6.** *Let $M \in \mathcal{M}_k^{eff}$. The presheaf $\omega_M$ can be uniquely extended to a pcdh-sheaf $\omega_M''$ on $\mathcal{P}_k$.*

*Proof.* From V.10(1) and the blow-up formula for Chow groups we deduce that $\omega_M$ is a *pcdh*-sheaf on $\mathcal{V}_k$. The result now follows from the first claim in V.10. □

**Lemma V.7.** *Let $M \in \mathcal{M}_k^{eff}$. We have $\omega_M''|_{\mathcal{V}_k'} \cong \omega_M'$.*

*Proof.* We will show that $\omega_M'$ extends uniquely to a *pcdh*-sheaf on $\mathcal{P}_k$. Since $\omega_M'|_{\mathcal{V}_k} \cong \omega_M$, V.6 shows that this extension is given $\omega_M''$. In particular, we have $\omega_M''|_{\mathcal{V}_k'} \cong \omega_M'$.

From the first statement in V.10, it suffices to show that $\omega_M'$ is a *pcdh*-sheaf on $\mathcal{V}_k'$. To do so, we use V.10(2). Let $X \in \mathcal{V}_k$ and $G$ a finite group acting on $X$. Let $Z \subset X$ be a smooth closed subscheme globally invariant under $G$. Let $\tilde{X}$ be the blow-up of $X$ along $Z$ and let $E$ be the exceptional divisor. We need to show that

$$\omega_M'(X/G) \simeq \ker\{\omega_M'(\tilde{X}/G) \oplus \omega_M'(Z/G) \to \omega_M'(E/G)\}.$$

This is equivalent to

$$\omega_M(X)^G \simeq \ker\{\omega_M(\tilde{X})^G \oplus \omega_M(Z)^G \to \omega_M(E)^G\}.$$

This is true by the blow-up formula for Chow groups and the exactness of the functor $(-)^G$ on $\mathbb{Q}[G]$-modules. □

**Lemma V.8.** *Let $M \in \mathcal{M}_k^{eff}$. Then $\omega''_M|_{\mathcal{N}_k}$ is an $fh$-sheaf.*

*Proof.* Let $X = Y/G$ with $Y \in \mathcal{N}_k$ and $G$ a finite group. We claim that $\omega''_M(Y)^G \cong \omega''_M(X)$. When $Y$ is smooth, this is true by V.5 and V.7. In general, we will prove this by induction on the dimension of $Y$ and we will no longer assume that $Y$ is normal. (However, it is convenient to assume that $Y$ is reduced.) If $Y$ has dimension zero then $Y$ is smooth and the result is known. Assume that $dim(Y) = d > 0$. By $G$-equivariant resolution of singularities there is a blow-up square

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "h"] & Y' \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

such that $Y'$ is smooth, $Z \subset Y$ is a nowhere dense closed subscheme which is invariant under the action of $G$ and such that $Y' - E \simeq Y - Z$. Taking quotients by $G$ gives the following blow-up square

$$\begin{tikzcd} E/G \arrow[r] \arrow[d] & Y'/G \arrow[d] \\ Z/G \arrow[r] & Y/G. \end{tikzcd}$$

Using induction on dimension and the fact that $\omega''_M$ is a *pcdh*-sheaf, we are left to show that $\omega''_M(Y')^G \simeq \omega''_M(Y'/G)$. This follows from V.5 and V.7. $\square$

*Proof of Theorem V.1.* It remains to show that the functor is full. Let $M, N \in \mathcal{M}_k^{eff}$. Since every effective motive is a direct summand of the motive of a smooth and projective variety, we may assume that $M = M(X)$ and $N = M(Y)$ for $X, Y \in \mathcal{V}_k$. Let $f : \omega_{M(X)} \to \omega_{M(Y)}$ be a morphism of presheaves. As in the proof of V.3, there is an associated morphism of Chow motives $f_X(id_{M(X)}) : M(X) \to M(Y)$. For $Z \in \mathcal{V}_k$ and $c \in \omega_{M(X)}(Z)$, we need to show that $f_X(id_{M(X)}) \circ c = f_Z(c) \in \omega_{M(Y)}(Z)$.

By V.7 the morphism $f$ can be uniquely extended to a morphism $f'' : \omega''_{M(X)} \to \omega''_{M(Y)}$ of $pcdh$-sheaves on $\mathcal{P}_k$. Moreover, by V.8, the restriction of $f''$ to $\mathcal{N}_k$ is a morphism of $fh$-sheaves. By [5, Prop 2.2.6] (see also [57]), any $fh$-sheaf has canonical transfers and $f''|_{\mathcal{N}_k}$ commutes with them. Now $c \in \omega_{M(X)}(Z) = CH^*(Z \times X)$ is the class of a finite correspondence $\gamma \in Cor(Z, X)$ and $c = \omega'_{M(X)}(\gamma)(id_{M(X)})$. (This follows from [47, Corollary 19.2] and the property that $C_*\mathbb{Q}_{tr}(X)$ is fibrant with respect to the projective motivic model structure; this property holds because $X$ is proper, see [6, Cor. 1.1.8].) Thus, we have:

$$f_Z(c) = f'_Z(\omega'_{M(X)}(\gamma)(id_{M(X)})) = \omega'_{M(Y)}(\gamma)(f'_X(id_{M(X)})) = f_X(id_{M(X)}) \circ c.$$

This completes the proof. □

### 5.2 Appendix B

Let $C'$ be a category and $\tau'$ a Grothendieck topology on $C'$. Given a functor $u : C \to C'$ there is an induced topology $\tau$ on $C$. (For the definition of the induced topology, we refer the reader to [3, III 3.1].)

**Proposition V.9.** *Assume that $u : C \hookrightarrow C'$ is fully faithful and that every object of $C'$ can be covered, with respect to the topology $\tau'$, by objects in $u(C)$. Let $X \in C$ and $R \subset X$ be a sub-presheaf of $X$. Then, the following conditions are equivalent:*

1. *$R \subset X$ is a covering sieve for $\tau$.*
2. *There exists a family $(X_i \to X)_{i \in I}$ such that*
   (a) *$R \supset Image(\coprod_i X_i \to X)$;*
   (b) *$(u(X_i) \to u(X))_{i \in I}$ is a covering family for $\tau'$.*

*Moreover $u_* : Shv(C') \to Shv(C)$ is an equivalence of categories.*

*Proof.* The last assertion is just [3, Théorème III.4.1].

(1) $\Rightarrow$ (2): Suppose $R \subset X$ is a covering sieve for $\tau$. Then $u^*(R) \to u(X)$ is a bicovering morphism for $\tau'$, i.e., induces an isomorphism on the associated sheaves (see [3, Définition I.5.1, Définition II.5.2 and Proposition III. 1.2] where $u^*$ was denoted by $u_!$ which is not so standard nowadays). Since $C$ contains a generating set of objects for $\tau'$, there is a covering family of the form $(u(X_i) \to u(X))_{i \in I}$ for the topology $\tau'$ and a dotted arrow as below

$$\begin{tikzcd} & u^*(R) \arrow[d] \\ \coprod_i u(X_i) \arrow[r] \arrow[ur, dotted] & u(X) \end{tikzcd}$$

making the triangle commutative.

Now, recall that for $U' \in C'$, one has

$$u^*(R)(U') = \underset{(V,\, U' \to u(V)) \in U' \backslash C}{colim} R(V)$$

where $U' \backslash C$ is the comma category. Using the fact that $u : C \hookrightarrow C'$ is fully faithful, we see that for $U' = u(U)$ the category $u(U) \backslash C$ has an initial object given by $(U, id : u(U) = u(U))$. It follows that $R(U) \simeq u^*(R)(u(U))$ which can be also written as $R \simeq u_* u^*(R)$. In particular, the maps of presheaves $u(X_i) \to u^*(R)$ are uniquely induced by maps of presheaves $X_i \to R$. This shows that $R$ contains the image of the morphism of presheaves $\coprod_i X_i \to X$.

(2) $\Rightarrow$ (1): Now suppose that condition (2) is satisfied. We must show that $u^*(R) \to u(X)$ is a bicovering morphism of presheaves for $\tau'$, i.e., that $a_{\tau'}(u^*(R)) \to a_{\tau'}(u(X))$ is an isomorphism where $a_{\tau'}$ is the "associated $\tau'$-sheaf" functor.

Since the surjective morphism of sheaves $a_{\tau'}(\coprod_i u(X_i)) \to a_{\tau'}(u(X))$ factors through $a_{\tau'}(u^*(R))$, the surjectivity of $a_{\tau'}(u^*(R)) \to a_{\tau'}(u(X))$ is clear. Since every object of $C'$ can be covered by objects in $u(C)$, to prove injectivity it suffices to show that

$u^*(R)(u(U)) \to u(X)(u(U))$ is injective for all $u \in C$. From the proof of the implication (1) $\Rightarrow$ (2), we know that this map is nothing but the inclusion $R(U) \hookrightarrow X(U)$. This finishes the proof. $\square$

The *pcdh* topology on $\mathcal{P}_k$ induces topologies on $\mathcal{V}_k$ and $\mathcal{V}'_k$ which we also call *pcdh*. The next corollary gives a description of these topologies.

**Corollary V.10.** *The categories of pcdh-sheaves on $\mathcal{V}_k$ and $\mathcal{V}'_k$ are equivalent to the category of pcdh-sheaves on $\mathcal{P}_k$. Moreover, pcdh-sheaves on $\mathcal{V}_k$ and $\mathcal{V}'_k$ can be characterized as follows.*

1. *A presheaf $F$ on $\mathcal{V}_k$ such that $F(\emptyset) = 0$ is a pcdh-sheaf if and only if for every smooth and projective variety $X$, and every closed and smooth subscheme $Z \subset X$, one has*
$$F(X) \simeq ker\{F(\tilde{X}) \oplus F(Z) \to F(E)\}$$
*where $\tilde{X}$ is the blow-up of $X$ in $Z$ and $E \subset \tilde{X}$ is the exceptional divisor.*

2. *A presheaf $F$ on $\mathcal{V}'_k$ such that $F(\emptyset) = 0$ is a pcdh-sheaf if and only if for every smooth and projective variety $X$ together with an action of a finite group $G$, and every closed and smooth subscheme $Z \subset X$ globally invariant under the action of $G$, one has*
$$F(X/G) \simeq ker\{F(\tilde{X}/G) \oplus F(Z/G) \to F(E/G)\}$$
*where $\tilde{X}$ is the blow-up of $X$ in $Z$ and $E \subset \tilde{X}$ is the exceptional divisor.*

*Proof.* By Hironaka's resolution of singularities, every projective variety can be covered (with respect to the *pcdh* topology) by smooth and projective varieties, i.e., by objects in the subcategory $\mathcal{V}_k$ (and hence $\mathcal{V}'_k$). Thus, the first claim follows from [3, Théorème III.4.1].

Next, we only treat (2) as the verification of (1) is similar and in fact easier. The condition in (2) is necessary for $F$ to be a *pcdh*-sheaf as $(\tilde{X}/G \to X/G,\ Z/G \to X/G)$ is a *pcdh*-cover. Hence we only need to show that the condition is sufficient.

Let $X/G \in \mathcal{V}'_k$ where $X$ is a smooth and projective variety and $G$ is a finite group acting on $X$. It suffices to show that $F(X/G) \simeq Colim_{R\subset(X/G)}\ F(R)$ where $R \subset (X/G)$ varies among covering sieves for the *pcdh*-topology on $\mathcal{V}'_k$. We will prove a more precise statement namely: any covering sieve $R \subset (X/G)$ can be refined into a covering sieve $R' \subset (X/G)$ such that $F(X/G) \simeq F(R')$.

By V.9, there exists a *pcdh*-cover $(Y_i \to (X/G))_i$ with $Y_i \in \mathcal{V}'_k$ and such that $R \supset Image(\coprod_i Y_i \to (X/G))$. Using equivariant resolution of singularities, we may find a sequence of equivariant blow-ups in smooth centers $Z_i \subset X_i$:

$$X_n \to \cdots \to X_1 \to X_0 = X$$

such that the covering family

$$(X_n/G \to X/G,\ Z_{n-1}/G \to X/G,\ \cdots,\ Z_0/G \to X/G) \tag{5.2.1}$$

is a refinement of the sieve $R$. Using induction and the property satisfied by $F$ from (2), we see that

$$F(X/G) \simeq ker\{F(X_n/G) \oplus F(Z_{n-1}/G) \oplus \cdots \oplus F(Z_0/G) \tag{5.2.2}$$

$$\longrightarrow F(E_n/G) \oplus \cdots \oplus F(E_1/G)\}$$

where $E_i \subset X_i$ is the exceptional divisor of the blow-up with center $Z_{i-1}$. It is easy to deduce from (5.2.2) that $F(X/G) \simeq F(R')$ when $R' \subset (X/G)$ is the image of the covering family (5.2.1). □

# BIBLIOGRAPHY

# BIBLIOGRAPHY


[1] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*, Annales de l'institut Fourier, 58 no. 4 (2008), p. 1057-1091.

[2] D. Arapura, A. Dhillon, *The motive of the moduli stack of $G$-bundles over the universal curve*, Indian academy of Sciences. Proceedings. Mathematical Sciences 118 (2008).

[3] M. Artin, A. Grothendieck and J.L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Séminaire de Géométrie Algébrique du Bois Marie, Tome 1, 1963/64.

[4] M. Artin, *Versal deformations and algebraic stacks*, Inventiones Mathematicae 27: p. 165-189.

[5] J. Ayoub, *Motifs des variétés analytiques rigides*, Preprint, http://user.math.uzh.ch/ayoub/

[6] J. Ayoub, *L'Algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I*, Preprint, http://user.math.uzh.ch/ayoub/

[7] J. Ayoub, *L'Algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II*, Preprint, http://user.math.uzh.ch/ayoub/

[8] J. Ayoub, *La réalisation étale et le opérations de Grothendieck*, Preprint, http://user.math.uzh.ch/ayoub/

[9] V. N. Aznar , S. D. B. Rollin, *On the motive of a quotient variety*, Collect. Math. 49, (1998), p. 203-226.

[10] K. Behrend, Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. 85 No. 1 (1996) p. 1-60.

[11] K. Behrand, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. 127 No. 3 (1997) p. 601-617.

[12] K. Behrand, A. Dhillon, *On the Motive of the stack of bundles.* http://arxiv.org/abs/math/0512640v1

[13] M. V. Bondarko, *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura*, J. of the Inst. of Math. of Jussieu, v.8 (2009), no. 1, p. 39-97.

[14] A. K. Bousfield, D.M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304. Springer-Verlag, 1972.

[15] U. Choudhury, *Motives of Deligne-Mumford Stacks.* Advances in Mathematics 231 (2012), pp. 3094-3117.

[16] J. Skowera, U. Choudhury, *Motivic decomposition for relative geometrically cellular stacks*, preprint, (submitted). http://www.math.uzh.ch/index.php?id=assi-publikationen&key1=4731.

[17] U. Choudhury, *On a conjecture of Morel*, preprint, (submitted). arXiv:1209.0484v1 [math.AG]

[18] B. Conrad, *The Keel-Mori theorem via stacks.*

[19] D. Dugger, S. Hollander and D. C. Isaksen, *Hypercovers and simpilcial presheaves*, Mathematical Proceedings of the Cambridge Philosophical Society, 136, pp 9-51 doi:10.1017/S0305004103007175

[20] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math. 131 No. 3 (1998) 595-634.

[21] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2, Berlin, New York: Springer-Verlag, ISBN 978-3-540-62046-4; 978-0-387-98549-7, MR1644323.

[22] H. Gillet, C. Soulé, *Motivic weight complexes for arithmetic varieties*, Journal of Algebra, Volume 322, Issue 9, 2009, 3088-3141.

[23] P. Gille, *Le problème de Kneser-Tits.* Astérisque 326 (2009), 39-82

[24] H. Gillet, *Intersection theory on algebraic stacks and* $\mathbb{Q}$*-varieties*; Journal of Pure and Applied Algebra 34 (1984) p. 193-240.

[25] P. G. Goerss , J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, 174 (1999). Birkhauser.

[26] M. Raynaud , L. Gruson; *Critères de platitude et de projectivité*, Inventiones math. 13, p. 1-89 (1971).

[27] S. Habibi, M.E. Arasteh Rad, *On the motive of* $G$*-bundles*, arXiv:1112.4110v3.

[28] P. S. Hirschhorn, *Model categories and their localizations.* volume 99 of Mathematical Surveys and Monographs , American Mathematical Society, 2009,

[29] S. Hollander, *A Homotopy Theory for Stacks*, Israel Journal of Math. 163 (2008), 93-124.

[30] J.-L. Colliot-Thelene, R.T. Hoobler, B. Kahn. *The Bloch-Ogus-Gabber theorem.* Fields Institute for Research in Mathematical Sciences Communications Series 16, A.M.S., 1997, 31-94.

[31] A. Huber, *Realization of Voevodsky's motives*, Journal of Algebraic Geometry 9(4): 755-799, 2000.

[32] F. Ivorra, Réalisation l-adique des motifs triangulés géométriques I, Doc. Math. 12 (2007), p. 607-671

[33] F. Ivorra, Réalisation l-adique des motifs triangulés géométriques II, Math. Zeit. 265 (2010), p. 221-247.

[34] J. F. Jardine, *Simplicial Presheaves*, J. Pure Appl. Algebra, 47(1):35-87,1987.

[35] J. F. Jardine, *Stacks and the homotopy theory of simplcial sheaves*, Homology, Homotopy and Applications, vol.3(2), 2001, pp. 361-384.

[36] R. Joshua, *Higher Intersection theory on algebraic Stacks, I*, K-Theory, 27, no.2, (2002), 134-195.

[37] R. Joshua, *Higher Intersection Theory on Algebraic Stacks, II*, K-Theory, 27, no.3, (2002), 197-244.

[38] N. Karpneko, *Cohomology of relative cellular spaces and isotropic flag varieties*, St. Petersburg Math. J. 12 (2001), no. 1, 1-50.

[39] S. Keel, S. Mori, *Quotients by groupoids*, Annals of math. 145 (1997), pp. 193-213.

[40] S. Kelly, *Vanishing of negative $K$-theory in positive characteristic*, arXiv:1112.5206v3 [math.AG]

[41] D. Knuston, *Algebraic Spaces*, Springer Lecture Notes in Mathematics, 203, 1971.

[42] J. Kollar, *Lectures on Resolution of Singularities*, Annals of Mathematics Studies. Princeton University Press.

[43] J. Kollar, *Quotient spaces modulo algebraic groups*. The Annals of Mathematics Second Series, Vol. 145, No. 1 (Jan., 1997), pp. 33-79

[44] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. 138 No. 3 (1999) 495-536.

[45] A. Kresch, A. Vistoli, *On coverings of Deligne-Mumford Stacks and surjectivity of the Brauer map*, Bull. London Math Soc. 36 (2004) 188-192.

[46] G. Laumon, L. Moret-Bailly, Champs Algébriques. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 39, Berlin, New York: Springer-Verlag.

[47] C. Mazza, V. Voevodsky, C. A. Weibel, *Lecture notes on motivic cohomology*, available at http://math.rutgers.edu/~weibel/motiviclectures.html.

[48] F. Morel, V. Voevodsky, $\mathbb{A}^1$-*homotopy theory of schemes*, Publications Mathématiques de l'IHES 90 (1990) p. 45-143.

[49] F. Morel, $\mathbb{A}^1$-*algebraic topology over a field*. Lecture Notes in Mathematics 2052, Springer Verlag, to appear. http://www.mathematik.uni-muenchen.de/~morel/listepublications.html

[50] F. Morel,*On the Friedlander-Milnor conjecture for groups of small rank*. Current Developments in Mathematics Volume 2010 (2011), 45-93.

[51] F. Morel , The stable $\mathbb{A}^1$-connectivity theorems. K-theory, 2005, vol 35, pp 1-68.

[52] D. Mumford, *Picard Groups of Moduli Problems*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), 1965, p. 33-81.

[53] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathmatiques de l'IHES (36): 75109

[54] D. Rydh, *Existence and properties of geometric quotients*, http://www.math.kth.se/~dary/papers.html

[55] A. J. Scholl, *Classical motives*, Proc. Symp. Pure Math. 55.1, (1994), p. 163-187.

[56] J. Skowera, *Białynicki-Birula decomposition of Deligne-Mumford stacks*. To appear in the Proceedings of the American Mathematical Society.

[57] A. Suslin, V. Voevodsky, *Singular homology of abstract algebraic varieties*, Inventiones Mathematicae, 123, p. 61-94 (1996).

[58] B. Toen, *On motives of Deligne-Mumford stacks*, Int Math Res Notices (2000) 2000 (17): 909-928.

[59] B. Toen, *K-théorie et cohomologie des champs algébriques*. PhD thesis, June 1999. available at http://ens.math.univ-montp2.fr/~toen/note.html.

[60] A. Vistoli, *Intersection theory on algebraic stacks and their moduli spaces*, Invent. Math. 97 (1989) p. 613-669.

[61] V. Voevodsky, *Cohomological theory of presheaves with transfers*, Cycles, transfers, and motivic cohomology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, p. 87-137

[62] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic cohomology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, p. 188-238.

[63] C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press.

[64] M. Wendt, $\mathbb{A}^1$-*homotopy groups of Chevalley groups.* J. K-Theory 5 (2), 2010, pp. 245–287.